# Evolution of OAM entanglement in turbulence

F. Stef Roux[a], T. Wellens[b] and V. Shatokhin[b]

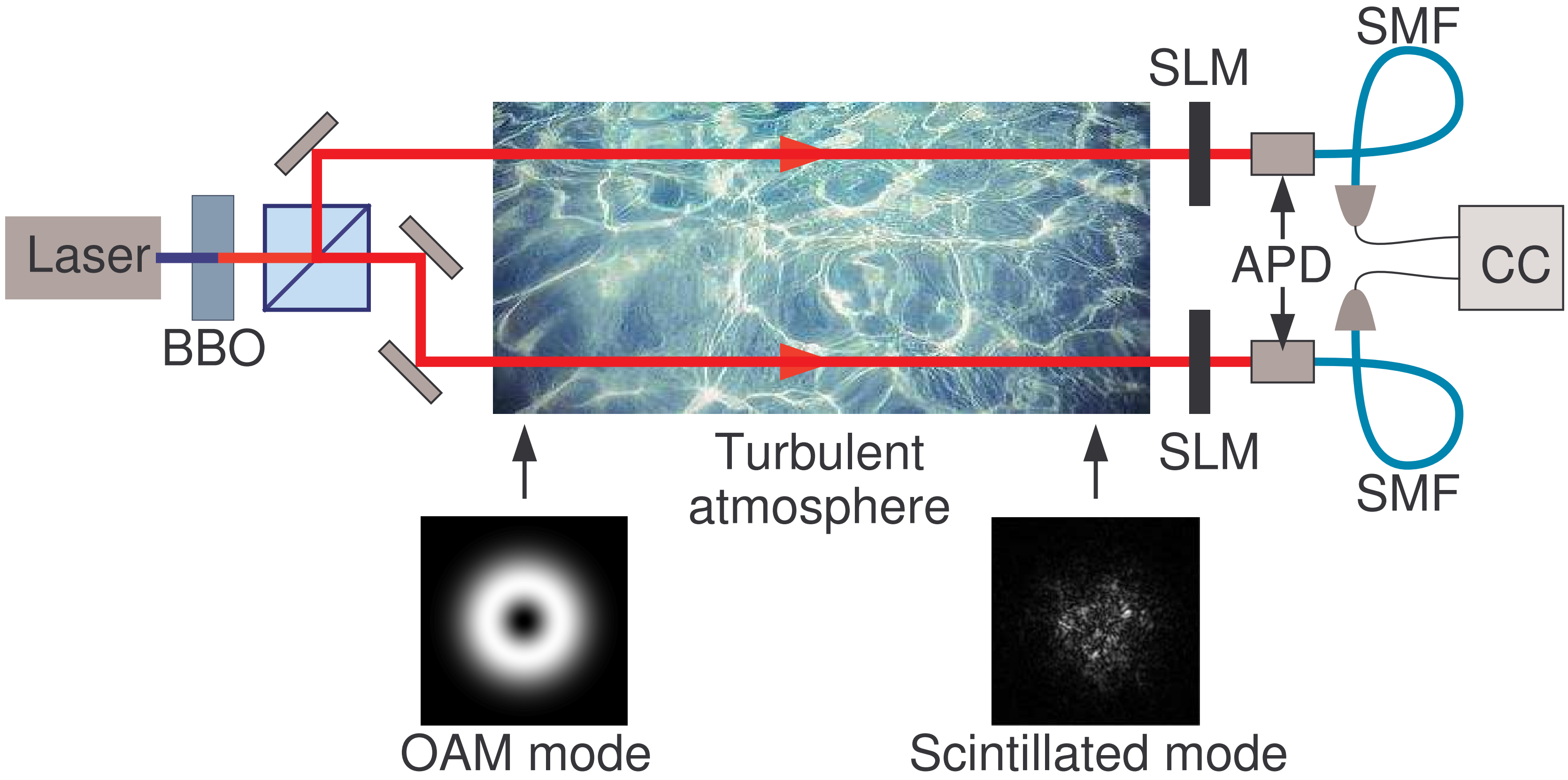
[a] CSIR National Laser Centre, South Africa
[b] Institute of Physics, Albert-Ludwigs University of Freiburg, Germany

Presented at the
*Third International Conference on Optical Angular Momentum,*
*City College of New York, New York City, USA*
*6 August 2015*

our future through science

# Free-space quantum communication

An orbital angular momentum (OAM) entangled photon pair is sent through the atmosphere and measured at the receiver.



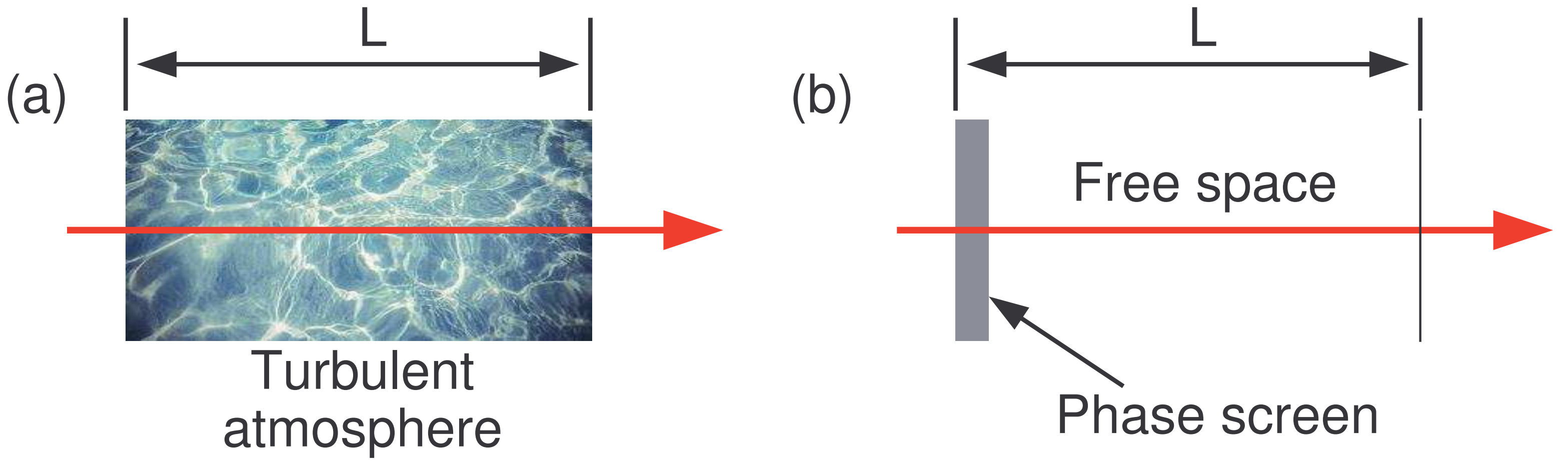

Turbulence distorts the OAM modes $\Rightarrow$ loss of entanglement

$\Rightarrow$ How to determine the evolution of a quantum state in turbulence?

# Single phase screen

Evolution of (scalar) photonic quantum states in turbulence

Single phase screen (SPS) approach (under weak scintillation): [a]




Ensemble averaged density matrix element:

$$\rho_{mn}(z) = \int E_m^*(\mathbf{r}_1) E_n(\mathbf{r}_2) \rho_0(\mathbf{r}_1, \mathbf{r}_2) \exp\left[-\frac{1}{2} D_\theta(\Delta r)\right] \mathrm{d}^2 r_1 \mathrm{d}^2 r_2$$

Modal basis functions: $E_n(\mathbf{r})$

Structure function: $D_\theta(\Delta r) = 6.88(|\mathbf{r}_1 - \mathbf{r}_2|/r_0)^{5/3}$

Fried parameter: $r_0 = 0.185(\lambda^2/C_n^2 z)^{3/5}$

$C_n^2$ is the structure constant; $\lambda$ is wavelength and $z$ is propagation distance

[a] C. Paterson, Phys. Rev. Lett. 94, 153901 (2005).

# OAM in turbulence (SPS approach)

Application of SPS approach:

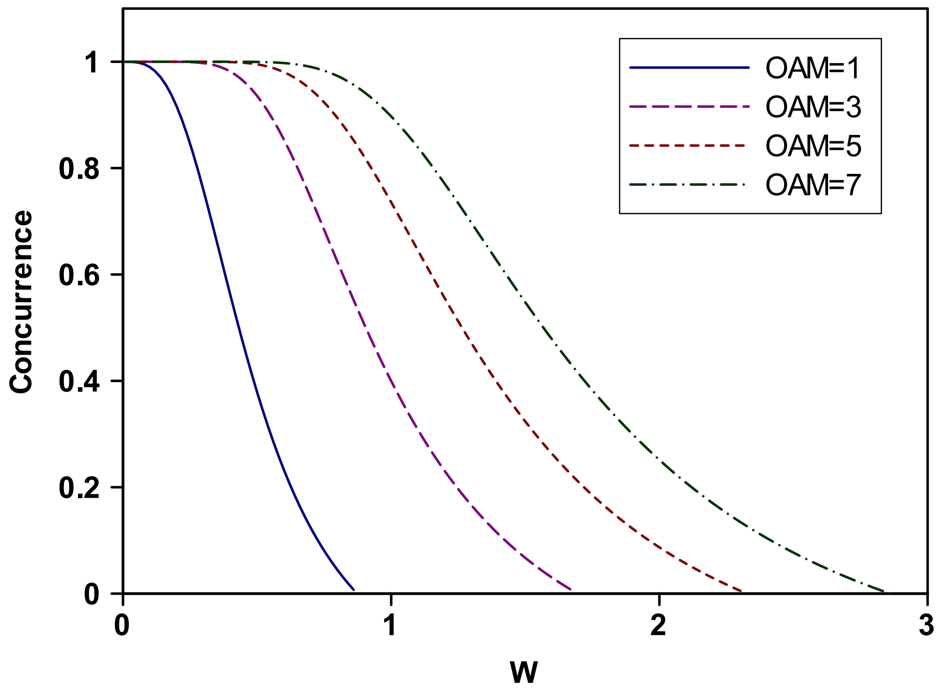
Decay of OAM entanglement in turbulence: [a]

Concurrence is a measure of qubit entanglement

$$\mathcal{W} = \frac{w_0}{r_0}$$

$w_0$ — beam radius

$r_0$ — Fried parameter




For example, for OAM=1:

$$\mathcal{C} = \frac{4(\chi+1)^2 - \chi^4}{4(\chi^2+\chi+1)^2}$$

where $\chi = 3.44\mathcal{W}^{5/3}$

[a] B. J. Smith and M. G. Raymer, Phys. Rev. A, 74, 062104 (2006)

# Weak scintillation limit

Consider scintillation strength vs normalized propagation distance

Rytov variance
(scintillation strength):

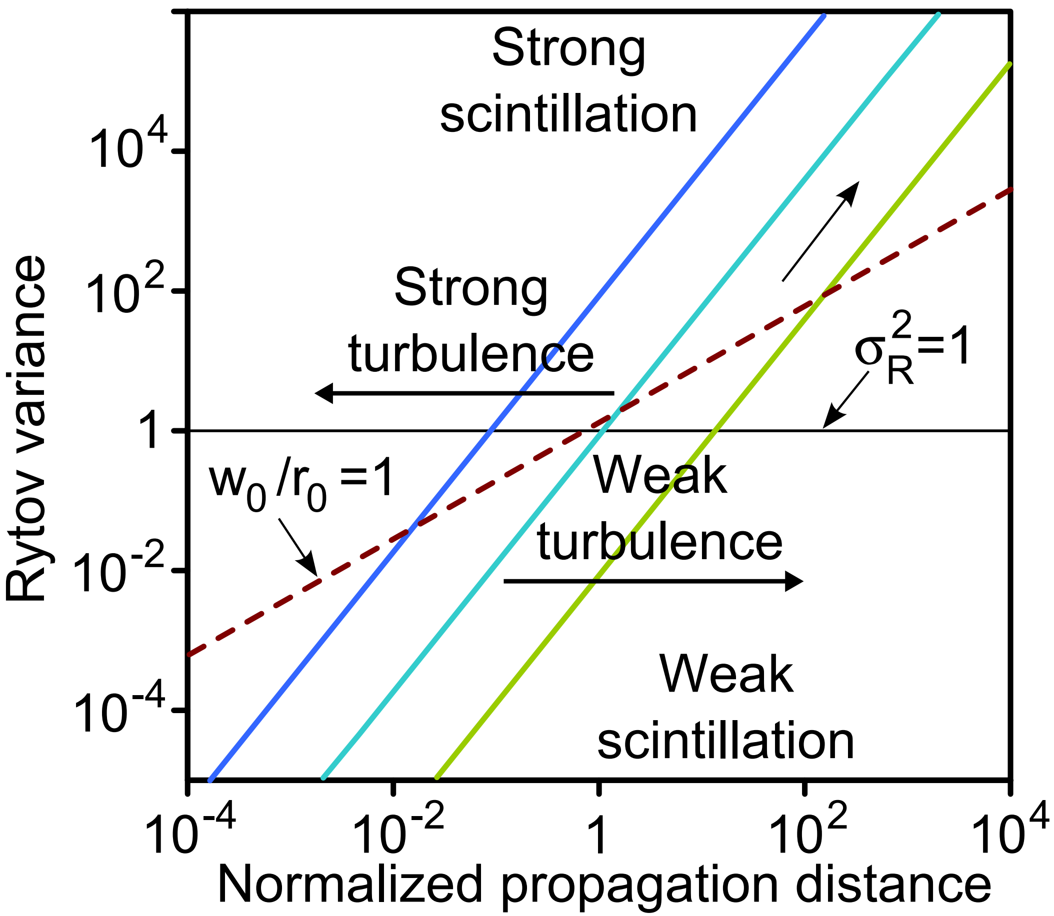
$$\sigma_R^2 = 1.23 C_n^2 k^{7/6} z^{11/6}$$

Using $t = z/z_R = z\lambda/\pi w_0^2$:

$$\sigma_R^2 = \frac{85.6 C_n^2 w_0^{11/3} t^{11/6}}{\lambda^3}$$

and using $\mathcal{W} = w_0/r_0$:

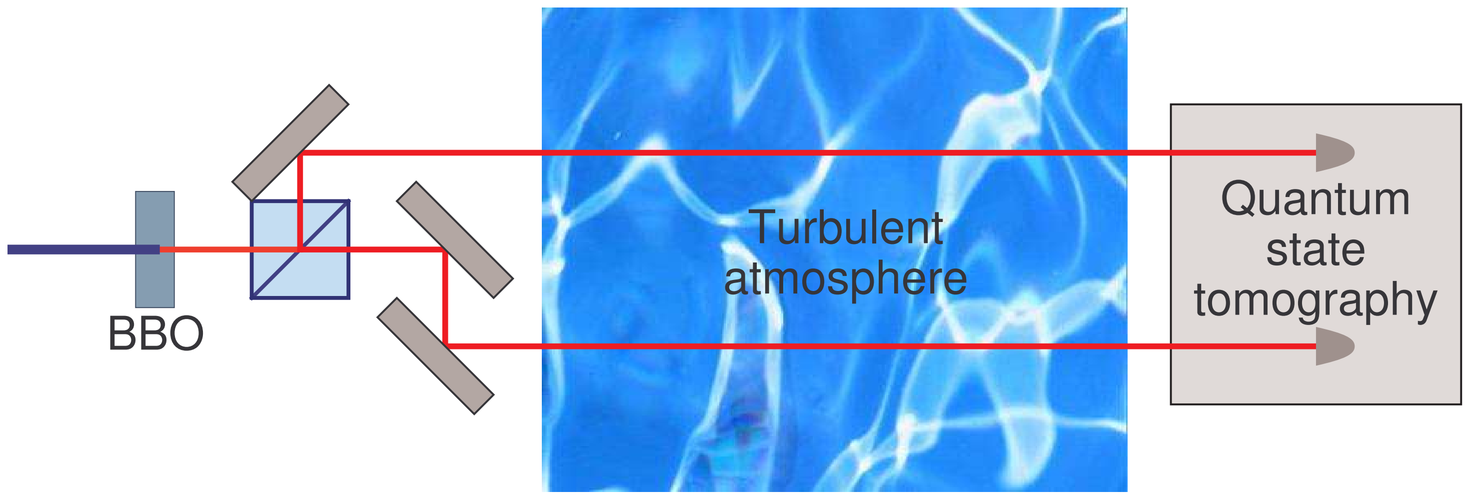
$$\sigma_R^2 = 1.64 \left(\frac{w_0}{r_0}\right)^{5/3} t^{5/6}$$

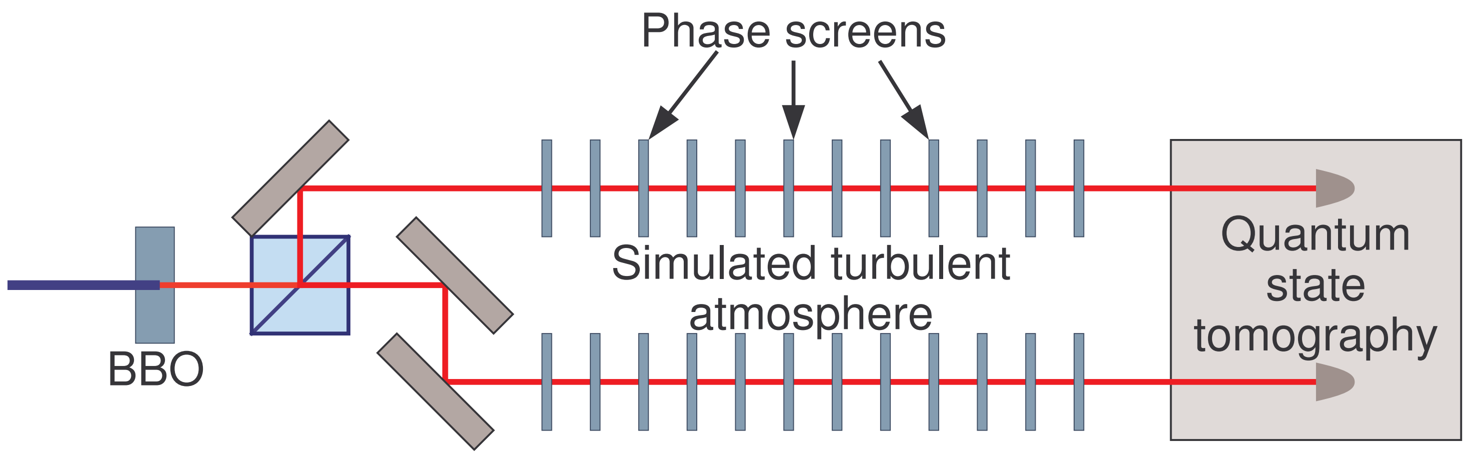



$\Rightarrow$ Weak scintillation for all $\mathcal{W}$ requires strong turbulence

# Multiple phase screen simulations

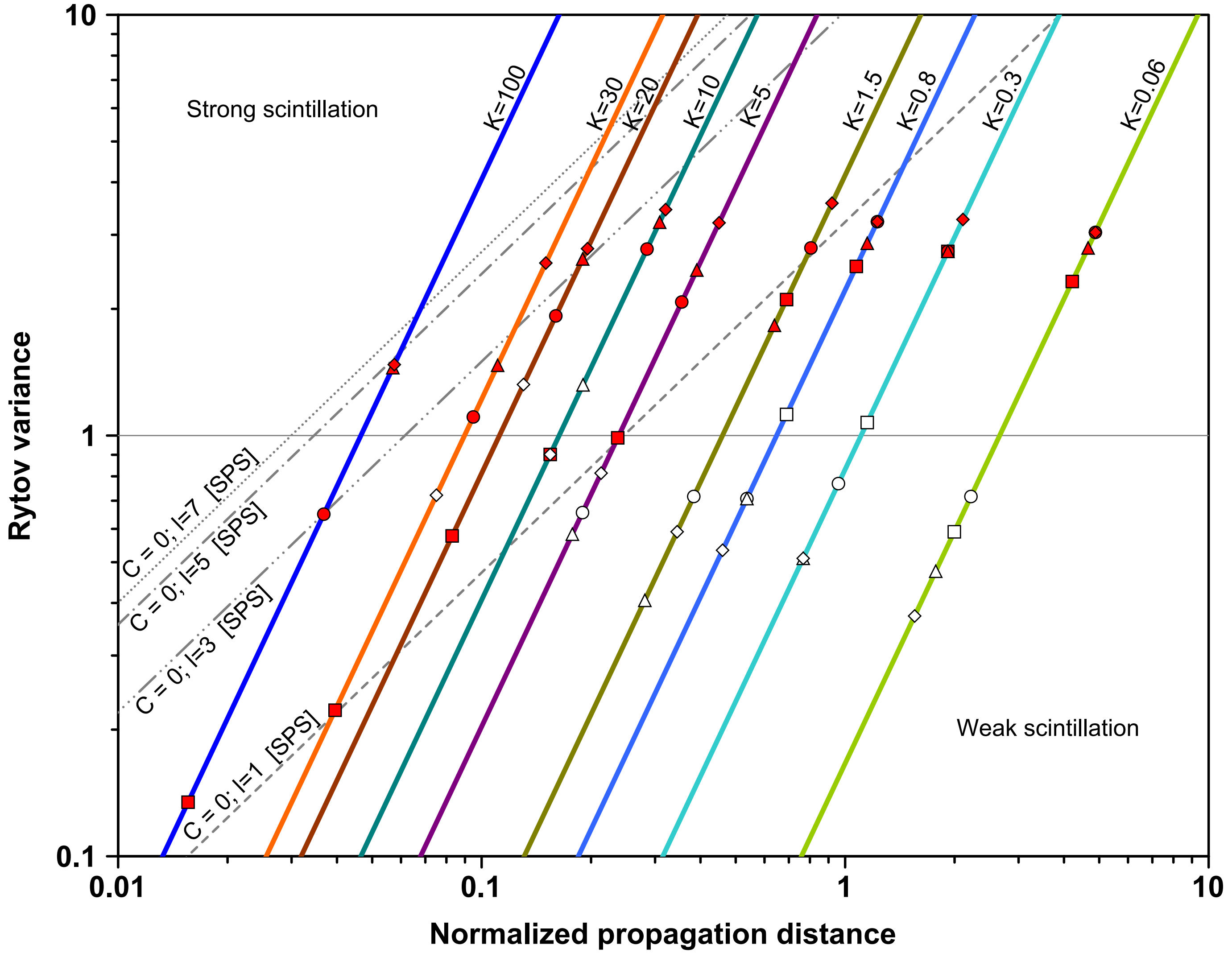
Simulate turbulent medium using multiple phase screens:



# Numerical simulation results

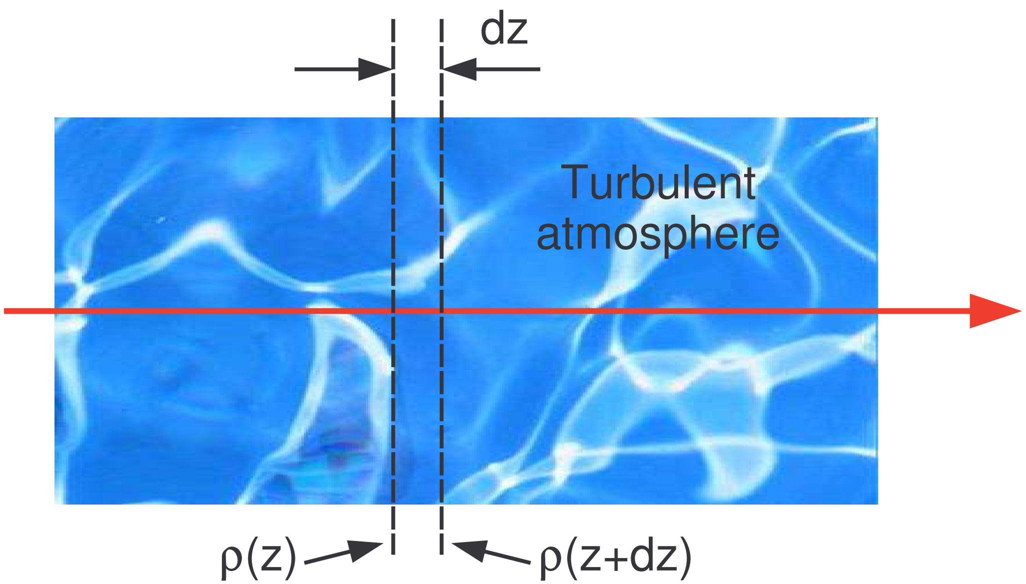
Entanglement ‘sudden death’ for input Bell-state in OAM basis [a]




[a] A. Hamadou Ibrahim, *et al.*, Phys. Rev. A, 90, 052115 (2014)

# Infinitesimal propagation (MPS approach)

Multiple phase screen (MPS) approach:




Infinitesimal propagation:
$\rho(z) \to \rho(z + \delta z)$

Using paraxial wave equation in an inhomogenous medium:

$$\nabla_T^2 g(\mathbf{x}) - i2k\partial_z g(\mathbf{x}) + 2k^2 \tilde{n}(\mathbf{x}) g(\mathbf{x}) = 0$$

In the (2D transverse spatial) Fourier domain:

$$G(\mathbf{a}, z + \delta z) = G(\mathbf{a}, z) + i\pi\lambda\delta z \left[ |\mathbf{a}|^2 G(\mathbf{a}, z) - \frac{k^2}{2\pi^2} N(\mathbf{a}, z) \star G(\mathbf{a}, z) \right]$$

$G(\mathbf{a}, z)$ — transverse Fourier transformed field (angular spectrum)
$N(\mathbf{a}, z)$ — transverse Fourier transformed refractive index fluctuations
$\mathbf{a}$ — spatial frequency vector ($\mathbf{k} = 2\pi\mathbf{a}$)

# Infinitesimal propagation equation

Expand to second order in fluctuations and evaluate ensemble average, using Markov approximation [$N(\mathbf{a},z)$ is delta-correlated in $z$].

Infinitesimal propagation equation for a single photon: [a]

$$\begin{aligned}\partial_z \rho(\mathbf{a}_1,\mathbf{a}_2,z) &= i\pi\lambda\rho(\mathbf{a}_1,\mathbf{a}_2,z)\left(|\mathbf{a}_1|^2-|\mathbf{a}_2|^2\right) \\ &\quad -k^2\int \Phi_0(\mathbf{u},0)\left[\rho(\mathbf{a}_1,\mathbf{a}_2,z)-\rho(\mathbf{a}_1-\mathbf{u},\mathbf{a}_2-\mathbf{u},z)\right]\ \mathrm{d}^2u\end{aligned}$$

$\rho(\mathbf{a}_1,\mathbf{a}_2,z)$ — single photon density function in plane wave basis

Density operator: $\hat{\rho}=\int |\mathbf{a}_1\rangle\,\rho(\mathbf{a}_1,\mathbf{a}_2,z)\,\langle\mathbf{a}_2|\ \mathrm{d}^2a_1\ \mathrm{d}^2a_2$

$\Phi_0(\mathbf{u},0)$ — power spectral density for the refractive index fluctuations

[a] F. S. Roux, Phys. Rev. A, 83, 053822 (2011);

—, J. Phys. A: Math. Theor., 47, 195302 (2014).

# Photon pairs (entanglement)

Infinitesimal propagation equation for photon pairs:

$$\begin{aligned}\partial_z \rho(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4,z) &= i\pi\lambda\rho(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4,z)(|\mathbf{a}_1|^2-|\mathbf{a}_2|^2+|\mathbf{a}_3|^2-|\mathbf{a}_4|^2) \\ &\quad -k^2\int \Phi_0(\mathbf{u},0)\left[2\rho(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4,z)\right. \\ &\quad -\rho(\mathbf{a}_1-\mathbf{u},\mathbf{a}_2-\mathbf{u},\mathbf{a}_3,\mathbf{a}_4,z) \\ &\quad \left.-\rho(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3-\mathbf{u},\mathbf{a}_4-\mathbf{u},z)\right]\ \mathrm{d}^2u\end{aligned}$$

$\rho(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4,z)$ — photon pair density 'matrix' in plane wave basis

Previously,[a] expanded $\rho$ in terms of LG basis
to obtain infinite set of coupled differential equations:

$$\partial_z\rho_{mnpq} = V_{mnrs}\rho_{rspq} + V_{pqrs}\rho_{mnrs} + L_{mnrs}\rho_{rspq} - L_T\rho_{mnpq}$$

which required truncation $\Rightarrow$ truncation problem!

[a] F. S. Roux, Phys. Rev. A, 83, 053822 (2011)

# Solving the IPE — without truncation

First for single photons — generalize for biphotons at the end:

1. Remove free-space term (quadratic phase factor):

$$\rho(\mathbf{a}_1, \mathbf{a}_2, z) = F(\mathbf{a}_1, \mathbf{a}_2, z) \exp[i\pi\lambda z(|\mathbf{a}_1|^2 - |\mathbf{a}_2|^2)]$$

2. Redefine coordinates i.t.o. sums and differences: $\mathbf{a}_{1,2} = \mathbf{a}_s \pm \mathbf{a}_d/2$

   so that: $F(\mathbf{a}_1, \mathbf{a}_2, z) = F(\mathbf{a}_s + \mathbf{a}_d/2, \mathbf{a}_s - \mathbf{a}_d/2, z) \equiv G(\mathbf{a}_s, \mathbf{a}_d, z)$

3. Inverse Fourier transform w.r.t. sum coordinates:

$$H(\mathbf{x}, \mathbf{a}_d, z) = \int G(\mathbf{a}_s, \mathbf{a}_d, z) \exp(-i2\pi\mathbf{a}_s \cdot \mathbf{x})\ \mathrm{d}^2 a_s$$

Resulting (single photon) equation:

$$\partial_z H(\mathbf{x}, \mathbf{a}_d, z) = -k^2 H(\mathbf{x}, \mathbf{a}_d, z) Q(\lambda z \mathbf{a}_d + \mathbf{x})$$

where: $$Q(\mathbf{x}) = \int \Phi_0(\mathbf{u}, 0)\,[1 - \exp(-i2\pi\mathbf{x} \cdot \mathbf{u})]\ \mathrm{d}^2 u$$

Solution: $$H(\mathbf{x}, \mathbf{a}_d, z) = H_0(\mathbf{x}, \mathbf{a}_d) \exp\left[-k^2 \int_0^z Q(\lambda z' \mathbf{a}_d + \mathbf{x})\ \mathrm{d}z'\right]$$

# Kolmogorov or quadratic structure function?

Q-integral: $$Q(\mathbf{x}) = \int \Phi_0(\mathbf{u}, 0) \left[1 - \exp(-i2\pi\mathbf{x}\cdot\mathbf{u})\right] \, \mathrm{d}^2 u$$

Kolmogorov spectral density: $$\Phi_0(\mathbf{u}, 0) = \frac{0.033 C_n^2}{(2\pi)^{2/3} |\mathbf{u}|^{11/3}}$$

($2\pi$-factor due to use of spatial frequency)

Result: $Q(\mathbf{x}) = 1.457 C_n^2 |\mathbf{x}|^{5/3}$

Quadratic structure function approximation: $5/3 \to 2$

Result: $$H(\mathbf{x}, \mathbf{a}_d, z) = H_0(\mathbf{x}, \mathbf{a}_d) \exp\left[\frac{-2\mathcal{K}_0 \Theta}{w_0^3} p(\mathbf{x}, \mathbf{a}_d, z)\right]$$

$\Theta = \lambda / \pi w_0$ — beam divergence angle

$w_0$ — beam radius

$$\mathcal{K}_0 = 2.9 \frac{\pi^3 C_n^2 w_0^{11/3}}{\lambda^3} = 2.9 \mathcal{K}$$

$$p(\mathbf{x}, \mathbf{a}_d, z) = |\mathbf{x}|^2 z + \mathbf{a}_d \cdot \mathbf{x} \, \lambda z^2 + \frac{1}{3} |\mathbf{a}_d|^2 \lambda^2 z^3$$

# Single photon state

Convert back to original variables

Evolving single photon state in turbulence:

$$\rho(\mathbf{a}_1,\mathbf{a}_2,t) = \frac{\pi w_0^2}{2\mathcal{K}_0 t}\int \rho_0(\mathbf{u}+\mathbf{a}_1,\mathbf{u}+\mathbf{a}_2)\exp\left\{-\pi^2 w_0^2\left[\frac{\mathcal{K}_0 t^3}{6}|\mathbf{a}_1-\mathbf{a}_2|^2 -\mathrm{i}t\ \mathbf{u}\cdot(\mathbf{a}_1-\mathbf{a}_2) - \mathrm{i}t\ (|\mathbf{a}_1|^2-|\mathbf{a}_2|^2) + \frac{|\mathbf{u}|^2}{2\mathcal{K}_0 t}\right]\right\}\ \mathrm{d}^2u$$

Normalized propagation distance: $t = \dfrac{z\lambda}{\pi w_0^2}$

$\rho_0(\mathbf{a}_1,\mathbf{a}_2)$ — input state at $t=0$

# Biphoton photon state (entanglement)

Evolving biphoton state in turbulence:

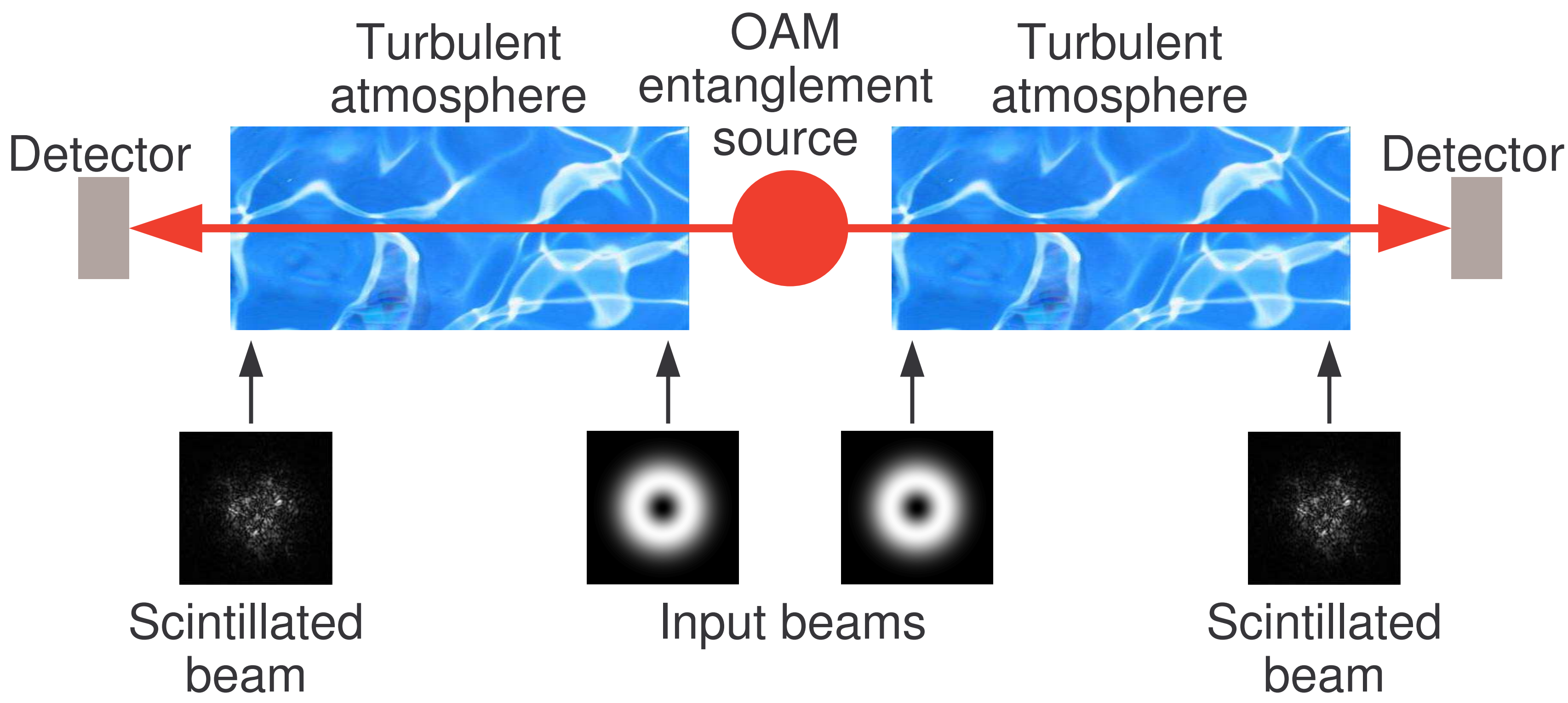
$$
\begin{aligned}
\rho(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4,t) \quad = \quad & \left(\frac{\pi w_0^2}{2\mathcal{K}_0 t}\right)^2 \int \rho_0(\mathbf{u}_1+\mathbf{a}_1,\mathbf{u}_1+\mathbf{a}_2,\mathbf{u}_2+\mathbf{a}_3,\mathbf{u}_2+\mathbf{a}_4) \\
& \times \exp\left\{-\pi^2 w_0^2 \left[\frac{\mathcal{K}_0 t^3}{6}|\mathbf{a}_1-\mathbf{a}_2|^2 - \mathrm{i}t\ \mathbf{u}_1\cdot(\mathbf{a}_1-\mathbf{a}_2)\right.\right. \\
& \left.\left. -\mathrm{i}t\ (|\mathbf{a}_1|^2-|\mathbf{a}_2|^2) + \frac{|\mathbf{u}_1|^2}{2\mathcal{K}_0 t}\right]\right\} \\
& \times \exp\left\{-\pi^2 w_0^2 \left[\frac{\mathcal{K}_0 t^3}{6}|\mathbf{a}_3-\mathbf{a}_4|^2 - \mathrm{i}t\ \mathbf{u}_2\cdot(\mathbf{a}_3-\mathbf{a}_4)\right.\right. \\
& \left.\left. -\mathrm{i}t\ (|\mathbf{a}_3|^2-|\mathbf{a}_4|^2) + \frac{|\mathbf{u}_2|^2}{2\mathcal{K}_0 t}\right]\right\}\ \mathrm{d}^2u_1\ \mathrm{d}^2u_2
\end{aligned}
$$

$\rho_0(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)$ — input state at $t=0$

# Evolution of OAM entanglement

As example, consider evolution of OAM entanglement due to turbulence for input Bell-state:

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}} \left( |\ell\rangle_A \, |-\ell\rangle_B - |-\ell\rangle_A \, |\ell\rangle_B \right)$$



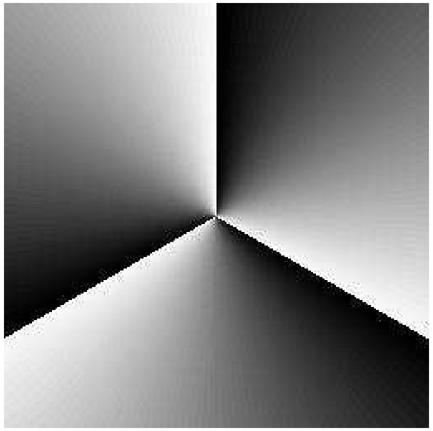
# Laguerre-Gauss (LG) modes

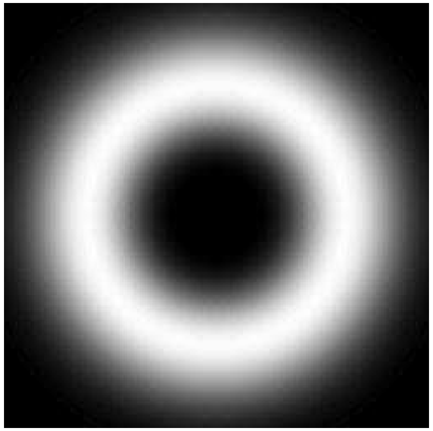
General solutions of the paraxial wave equation
in normalized polar coordinates:

$$M_{p,\ell}^{\mathrm{LG}}(r,\phi,t) = \mathcal{N}\frac{(1+it)^p}{(1-it)^{p+|\ell|+1}} r^{|\ell|}\exp(i\ell\phi) L_p^{|\ell|}\left(\frac{2r^2}{1+t^2}\right)\exp\left(\frac{-r^2}{1-it}\right)$$

$$r = \frac{\sqrt{x^2+y^2}}{w_0} \qquad t = \frac{z}{z_R} \qquad z_R = \frac{\pi w_0^2}{\lambda}$$

$$\mathcal{N} = \left[\frac{2^{1+|\ell|}p!}{\pi(p+|\ell|)!}\right]^{1/2}$$





$L_p^{|\ell|}$ — associate Laguerre polynomials
$p$ — radial mode index (non-negative integer)
$\ell$ — azymuthal index (signed integer)

Orbital angular momentum (OAM) per photon $= \ell\hbar$.

# LG mode generating function

Generating function for angular spectra of LG modes:

$$\mathcal{G} = \frac{\pi w_0}{1+\eta} \exp\left[\frac{i\pi(a \pm ib)w_0\mu - \pi^2(a^2+b^2)w_0^2\Omega(t,\eta)}{1+\eta}\right]$$

where ($\eta, \mu$) are generating parameters for ($p, \ell$)

and $\Omega(t,\eta) = 1 - it - (1+it)\eta$

To generate a particular LG mode's angular spectrum:

$$M_{p,\ell}^{\mathrm{LG}}(\mathbf{a}) = \mathcal{N}\left[\frac{1}{p!}\partial_\eta^p \partial_\mu^{|\ell|}\mathcal{G}\right]_{\eta,\mu=0}$$

# IPE calculations

Steps:

1. Produce input density matrix (Bell-state) i.t.o. generating function
2. Produce overlap function i.t.o. generating function
3. Evaluate all integrals
   $\Rightarrow$ generating function for density matrix elements
   having 8 generating parameters for azimuthal indices (assume $p = 0$).
4. Generate the density matrix for particular azimuthal indices
5. Calculated the concurrence
   $\Rightarrow$ concurrence as function of $t$ and $\mathcal{K}$.

$$\mathcal{K} = \frac{\pi^3 C_n^2 w_0^{11/3}}{\lambda^3}$$

# Weak scintillation limit

One can obtain the weak scintillation limit result from the IPE:

Weak scintillation condition: $\sigma_R^2 \lesssim 1$.

Express $\sigma_R^2$ i.t.o. $\mathcal{W}$ and $\mathcal{K}$, using definitions of $\mathcal{W}$ and $\mathcal{K}$.

Subsitute (into $\sigma_R^2$):

$$t \to \frac{0.59\mathcal{W}^{5/3}}{\mathcal{K}}$$

Result:

$$\sigma_R^2 = \frac{1.055\mathcal{W}^{55/18}}{\mathcal{K}^{5/6}}$$

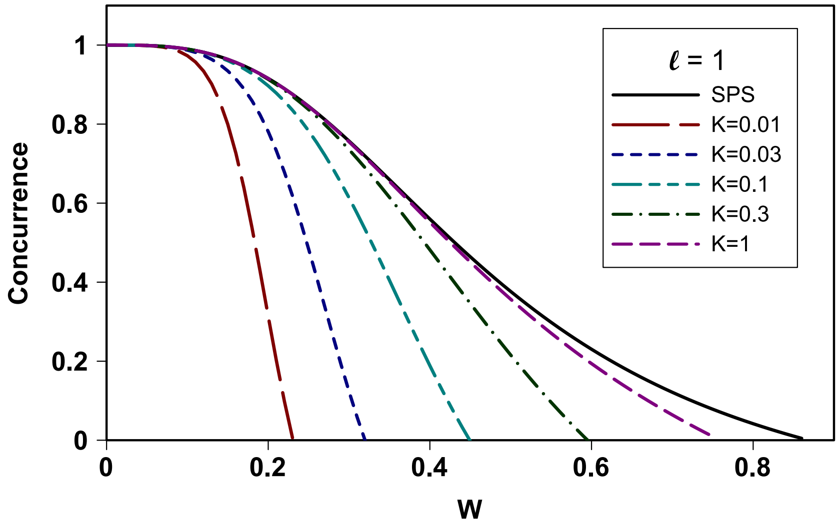
Conclusion: for weak scintillation limit ($\sigma_R^2 \to 0$) we need $\mathcal{K} \to \infty$.

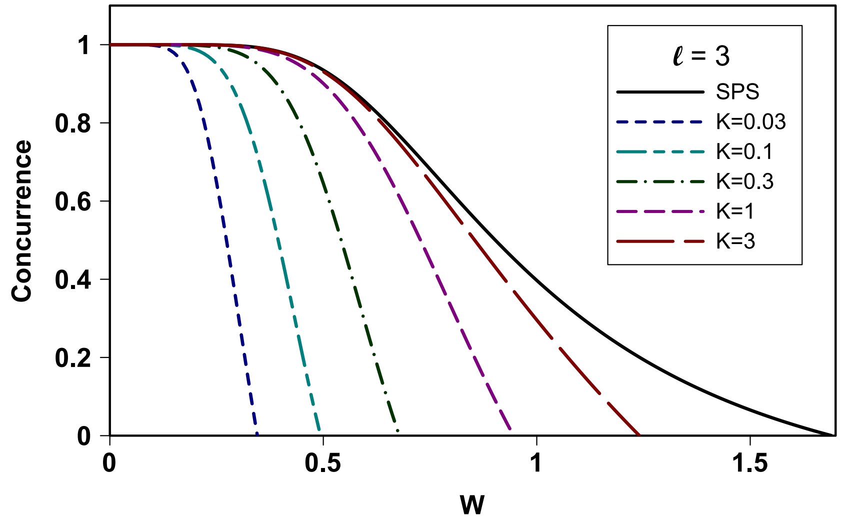
Applying this weak scintillation limit to IPE result, we obtain for $\ell = 1$:

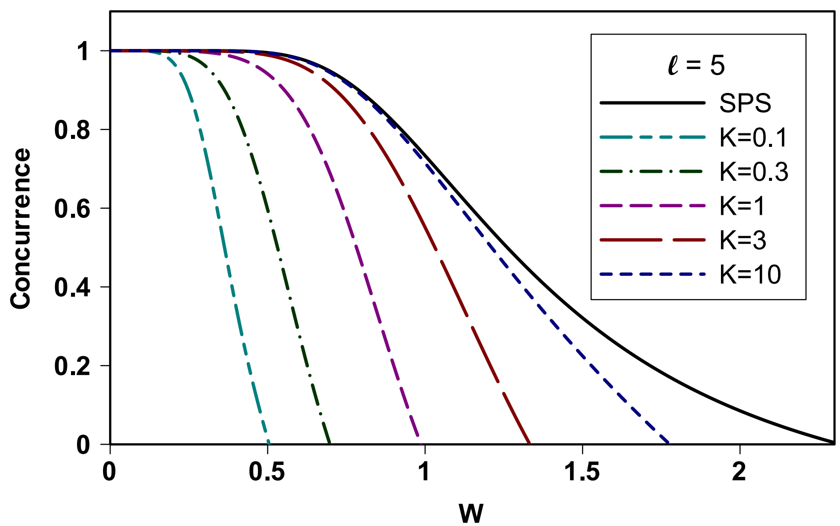
$$\mathcal{C} = \frac{4(\chi+1)^2 - \chi^4}{4(\chi^2+\chi+1)^2} \qquad \chi = 3.44\mathcal{W}^{5/3}$$

$\Rightarrow$ Identical to SPS result.

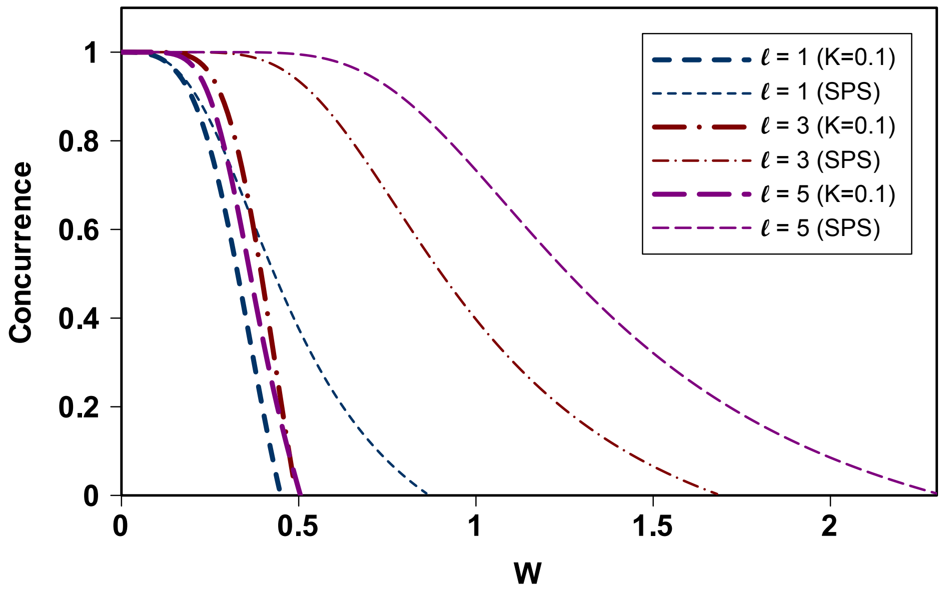
# OAM entanglement

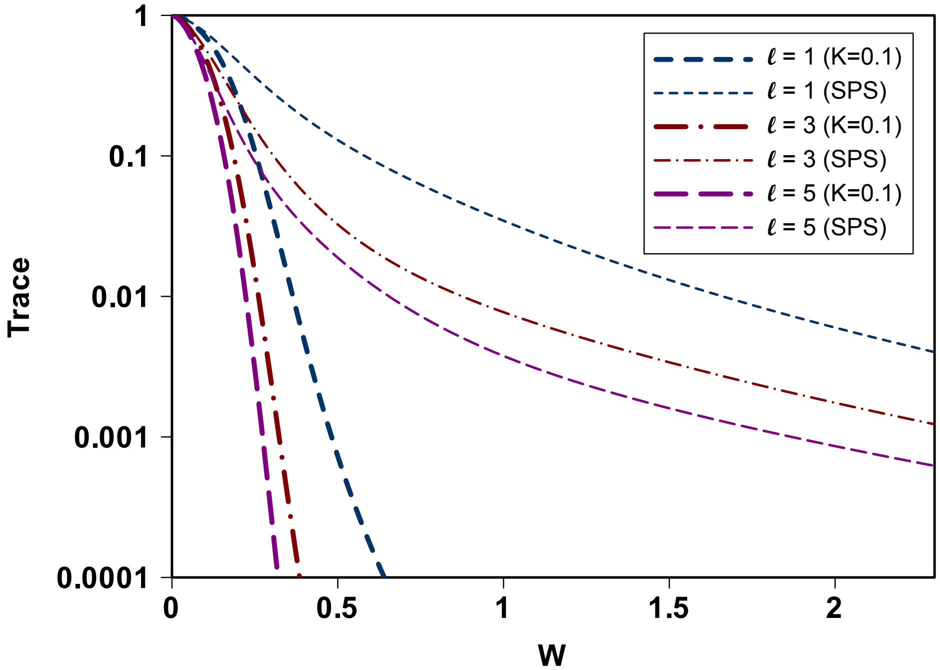


# Is larger OAM better?




$\Rightarrow$ no clear benefit in using higher OAM.

# Summary

- ▷ Turbulence distorts spatial modes $\Rightarrow$ loss of entanglement
- ▷ Investigate evolution of quantum states in turbulence
  - — Single phase screen (SPS) approach
  - — Multiple phase screen (MPS) approach
- ▷ Numerical simulations show where SPS breaks down
- ▷ Infinitesimal propagation equation (IPE) provides MPS approach
- ▷ Solve IPE without truncation in Fourier domain
- ▷ Application: OAM entanglement evolution in turbulence
- ▷ Allows us to investigate the effects of:
  - — weak turbulence
  - — higher OAM