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On the Resonant Behavior of Longitudinally Vibrating Accreting Rods

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Abstract—**The theory of accreting structures is a new and fast developing branch of analytical mechanics basing on the theory of partial differential and integral equations. In the present paper the authors analyze qualitative properties of accreting rods subjected to longitudinal vibrations. This problem is described it terms of the linear classical, Rayleigh-Love and Rayleigh-Bishop models. It is assumed that the rod is fixed at one end and free at the other end and its length is increasing. For solution of this problem we make a special change of variables which transforms the original equations into new non-autonomous equations. It is shown that these equations are hyperbolic and possess several interesting and important properties. First of all, the amplitudes of vibration of the rod are growing with time. For example, if the rod length is increasing proportionally to time the amplitudes are also growing proportionally to time. Secondly, if a particular mode is excited it excites other modes. In this case the mechanism of the modes excitation is asymmetric, which means that the low frequency modes possess higher amplitudes compared to the higher frequency modes. The physical explanation of these phenomena is proposed and discussed.**

Keywords— Longitudinal vibration of rods, accreting rod, growing rod, resonance.

INTRODUCTION

The word "accreting" means growing of a body in one or several dimensions. For example, let us consider the longitudinally vibrating rod, which is fixed at its left end and free at the right end, and assume that its length is increasing from the free end. Dynamics of this model is investigated in the present paper. There exist many different models describing the longitudinal vibration of slender rods. In the simplest case the lateral effects are neglected at the longitudinal vibration and the dynamics of such rod is described in terms of the wave equation. This is the classical model of the longitudinal vibration of the rod. In the more advanced Rayleigh–Love model the effect of lateral inertia of the longitudinally vibrating rod is taken into consideration. This model is described by the equation with mixed partial derivative of the fourth order. Another, the Rayleigh–Bishop model takes into account both lateral inertia and shear stresses effects and described by equations with mixed and x – derivatives of the fourth order. Development of these three models of the accreting longitudinally vibrating rods is described in the present paper. The boundary conditions of the accreting rods are described by the non-conventional time dependent expressions. For conversion of the boundary problems to the conventional formulations a special transformation is introduced. The tradeoff of this approach is a substantial sophistication of the equations of motion of the rods which become non-autonomous with variable coefficients. The corresponding partial differential equations are transformed into the systems of ordinary differential equations by the Kantorovich method. This method is demonstrated on the classical model of the vibrating rod which is accreting proportionally to time. The infinite system of ordinary differential equations describing the dynamics of this rod is truncated and solved numerically. It is shown that this rod demonstrates the resonance behaviour and increases its amplitude proportionally to time. To explain this behaviour we analyze the modelling equation of the Euler type which describes the dynamics of an insulated vibrating mode. It is shown that this equation has an exact solution which qualitatively describes the mode behaviour, i.e. the amplitude of vibrations is growing proportionally to time.

EQUATIONS AND BOUNDARY CONDITIONS OF THE ROD DESCRIBED BY THE CLASSICAL MODEL

In the classical case the longitudinal motions of the vibrating rod its dynamics is described by the wave equation:

$$\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}=F(t,x) \qquad (1)$$

where $u=u(t,x)$ is longitudinal displacement of the rod, $F(t,x)$ is exciting force and $c^2=\sqrt{E/\rho}$ is speed of the wave propagation. It is assumed that the left end of the rod is fixed and its right end is free. The process of the rod growth is realized by means of deposition of the material of the rod on its right end. Hence, the boundary conditions are:

$$\begin{aligned} &x=0: && u(t,0)=0 \\ &x=1+\varepsilon f(t): && \frac{\partial u}{\partial x}\big(t,1+\varepsilon f(t)\big)=0 \end{aligned} \qquad (2)$$

where ε is a small parameter proportional to the speed of growth of the rod.

To represent the boundary value problem (1) – (2) in the standard form it is necessary to use transformation $(t,x)\to(\tau,y)$:

$$t=\tau \qquad x=y\left[1+\varepsilon f(t)\right] \qquad (3)$$

In new parameters (τ,y) equation (1) is as follows:

$$\begin{aligned} &\frac{\partial^2\tilde u}{\partial\tau^2}-\frac{2\varepsilon y f'(\tau)}{1+\varepsilon f(\tau)}\frac{\partial^2\tilde u}{\partial\tau\partial y}-\frac{c^2-\varepsilon^2 y^2 [f'(\tau)]^2}{\left[1+\varepsilon f(\tau)\right]^2}\frac{\partial^2\tilde u}{\partial y^2} \\ &-\left\{\frac{\varepsilon y f''(\tau)}{1+\varepsilon f(\tau)}-\frac{2\varepsilon^2 y\left[f'(\tau)\right]^2}{\left[1+\varepsilon f(\tau)\right]^2}\right\}\frac{\partial\tilde u}{\partial y}=\tilde F(\tau,y) \end{aligned} \qquad (4)$$

where $f'(\tau)=\frac{df(\tau)}{d\tau}$, $f''(\tau)=\frac{d^2 f(\tau)}{d\tau^2}$, $\tilde u(\tau,y)=u\left[t=\tau,x=y\left(1+\varepsilon f(\tau)\right)\right]$, $\tilde F(\tau,y)=F\left[t=\tau,x=y\left(1+\varepsilon f(\tau)\right)\right]$. In this case boundary conditions (2) becomes:

$$y=0:\ \tilde u(\tau,0)=0, \quad y=1:\ \frac{\partial\tilde u}{\partial y}(\tau,1)=0 \qquad (5)$$

In the particular case $f(\tau)=\tau$ (linear growth of the rod) equation (4) is:

$$A^{(1)}\tilde u(\tau,y)=\tilde F(\tau,y) \qquad (6)$$

where the classical linear differential hyperbolic operator is:

$$A^{(1)}=\frac{\partial^2}{\partial\tau^2}-\frac{2\varepsilon y}{1+\varepsilon\tau}\frac{\partial^2}{\partial\tau\partial y}-\frac{c^2-\varepsilon^2y^2}{(1+\varepsilon\tau)^2}\frac{\partial^2}{\partial y^2}+\frac{2\varepsilon^2 y}{(1+\varepsilon\tau)^2}\frac{\partial}{\partial y}$$

(7)

THE RAYLEIGH-LOVE MODEL OF VIBRATING ROD

Original equation of the vibrating rod in this model is:

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} - a^2\frac{\partial^4 u}{\partial t^2 \partial x^2} = F(t,x) \qquad (8)$$

where $a^2 = \nu^2 I_p / S$, ν is the Poisson ratio, I_p is the polar moment of inertia and S is area of the cross-section of the rod. Transforming equation (8) by (3) we obtain:

$$\left[A^{(C)} - a^2 A^{(RL)}\right]\tilde{u}(t,y) = \tilde{F}(t,y) \qquad (9)$$

where

$$A^{(RL)} = \frac{1}{(1+\alpha t)^2}\left\{\frac{\partial^4}{\partial t^2 \partial y^2} - \frac{2\alpha}{1+\alpha t}\left[y\frac{\partial^4}{\partial t \partial y^3} - 2\frac{\partial^3}{\partial t \partial y^2}\right] + \frac{\alpha^2}{(1+\alpha t)^2}\left[y^2\frac{\partial^4}{\partial y^4} + 6y\frac{\partial^3}{\partial y^3} + 6\frac{\partial^2}{\partial y^2}\right]\right\}$$

(10)

THE RAYLEIGH-BISHOP MODEL OF VIBRATING ROD

Original equation of the vibrating rod in this model is:

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} - a^2\frac{\partial^4 u}{\partial t^2 \partial x^2} - b^2\frac{\partial^4 u}{\partial x^4} = F(t,x) \qquad (11)$$

where $b^2 = \nu^2 G I_p / \rho S$, G is the shear modulus of elasticity and ρ is the mass density of the rod. Transforming equation (11) by (3) we obtain:

$$\left[A^{(C)} - a^2 A^{(RL)} - b^2 A^{(RB)}\right]\tilde{u}(t,y) = \tilde{F}(t,y)$$

(12)

where

$$A^{(RB)} = \frac{1}{(1+\alpha t)^4}\frac{\partial^4}{\partial y^4} \qquad (13)$$

SOLUTION OF EQUATION (4) OF THE ROD IN THE CLASSICAL MODEL

The numerical solution is obtained using the Galerkin-Kantorovich method with base functions $\sin\left[\frac{(2k+1)\pi}{2}y\right]$, satisfying boundary conditions (5). The following representation of solution is used:

$$\tilde{u}(t,y) = \sum_{n=1}^{N} C_n(t)\sin\left[\frac{(2n-1)\pi}{2}y\right] \qquad (14)$$

where $C_m(\tau)$ are unknown functions of time. Furthermore we assume that in equation (4) $\tilde{F}(\tau, y) = 0$ (free vibrations). Let us substitute expression (14) into equation (4), multiply it by $2\sin\left[\frac{(2n-1)\pi}{2}y\right]$, where $n = 1, 2, \ldots, N$, and integrate the result over y in the limits from 0 to 1. As a result we obtain the system of coupled ordinary differential equations:

$$\frac{d^2C_m}{dt^2} + \frac{\omega_m^2(\tau)}{\left[1+\varepsilon f(\tau)\right]^2}C_m - \frac{\varepsilon f'(\tau)}{1+\varepsilon f(\tau)}\frac{dC_m}{dt}$$

$$+\frac{\varepsilon f'(\tau)}{1+\varepsilon f(\tau)}\sum_{\substack{n=1\\(n\neq m)}}^{N}\frac{(-1)^{m+n}\cdot(2n-1)^2}{(m-n)(m+n-1)}\frac{dC_n}{dt} +$$

$$\left[\frac{\varepsilon f'(\tau)}{1+\varepsilon f(\tau)}\right]^2\frac{(2m-1)^2}{2}\sum_{\substack{n=1\\(n\neq m)}}^{N}\frac{(-1)^{m+n+1}\cdot(2n-1)^2}{(m-n)^2(m+n-1)^2}C_n \approx 0$$

(15)

where $m = 1, 2, \ldots, N$ and

$$\omega_m^2(\tau) = \frac{c^2\pi^2(2m-1)^2}{4} - \frac{\varepsilon(1+\varepsilon f(\tau))f''(\tau)}{2} - \frac{\varepsilon^2\left\{\pi^2(2m-1)^2 - 6\right\}\left[f'(\tau)\right]^2}{12} \quad (16)$$

Let us consider a particular case of the linear growth $f(\tau) = \tau$ of the rod. In this case the system of ordinary differential equations (15) is rewritten as follows (we restrict our consideration by $N = 4$):

$$m=1:\ \left\{\frac{d^2C_1}{dt^2} - \frac{\varepsilon}{1+\varepsilon\tau}\frac{dC_1}{dt} + \frac{\omega_1^2}{(1+\varepsilon\tau)^2}C_1\right\}$$

$$+\frac{\varepsilon}{1+\varepsilon\tau}\left\{\frac{9}{2}\frac{dC_2}{dt} - \frac{25}{6}\frac{dC_3}{dt} + \frac{49}{12}\frac{dC_4}{dt}\right\} +$$

$$\left(\frac{\varepsilon}{1+\varepsilon\tau}\right)^2\left\{\frac{9}{8}C_2 - \frac{25}{72}C_3 + \frac{49}{288}C_4\right\} \approx 0$$

$$m=2:\ \left\{\frac{d^2C_2}{dt^2} - \frac{\varepsilon}{1+\varepsilon\tau}\frac{dC_2}{dt} + \frac{\omega_2^2}{(1+\varepsilon\tau)^2}C_2\right\}$$

$$+\frac{\varepsilon}{1+\varepsilon\tau}\left\{-\frac{1}{2}\frac{dC_1}{dt} + \frac{25}{4}\frac{dC_3}{dt} - \frac{49}{10}\frac{dC_4}{dt}\right\} +$$

$$\left(\frac{\varepsilon}{1+\varepsilon\tau}\right)^2\left\{\frac{9}{8}C_1 + \frac{225}{32}C_3 - \frac{441}{200}C_4\right\} \approx 0$$

(17)

$$m=3:\ \left\{\frac{d^2C_3}{dt^2}-\frac{\varepsilon}{1+\varepsilon\tau}\frac{dC_3}{dt}+\frac{\omega_3^2}{(1+\varepsilon\tau)^2}C_3\right\}$$

$$+\frac{\varepsilon}{1+\varepsilon\tau}\left\{\frac{1}{6}\frac{dC_1}{dt}-\frac{9}{4}\frac{dC_2}{dt}-\frac{49}{6}\frac{dC_4}{dt}\right\}+$$

$$\left(\frac{\varepsilon}{1+\varepsilon\tau}\right)^2\left\{-\frac{25}{72}C_1+\frac{225}{32}C_2+\frac{1225}{72}C_4\right\}=0$$

$$m=4:\ \left\{\frac{d^2C_4}{dt^2}-\frac{\varepsilon}{1+\varepsilon\tau}\frac{dC_4}{dt}+\frac{\omega_4^2}{(1+\varepsilon\tau)^2}C_4\right\}$$

$$+\frac{\varepsilon}{1+\varepsilon\tau}\left\{-\frac{1}{12}\frac{dC_1}{dt}+\frac{9}{10}\frac{dC_2}{dt}-\frac{25}{6}\frac{dC_3}{dt}\right\}+$$

$$\left(\frac{\varepsilon}{1+\varepsilon\tau}\right)^2\left\{\frac{49}{288}C_1-\frac{441}{200}C_2+\frac{1225}{72}C_3\right\}=0$$

where

$$\omega_1^2=\frac{1}{12}\left[3\pi^2c^2-\varepsilon^2\left(\pi^2-6\right)\right]$$

$$\omega_2^2=\frac{1}{4}\left[9\pi^2c^2-\varepsilon^2\left(3\pi^2-2\right)\right],\ \omega_3^2=\frac{1}{12}\left[75\pi^2c^2-\varepsilon^2\left(25\pi^2-6\right)\right],\ \omega_4^2=\frac{1}{12}\left[147\pi^2c^2-\varepsilon^2\left(49\pi^2-6\right)\right].$$

MODELLING EQUATION FOR THE VIBRATING ROD

It follows from system (17) that for all modes the following modelling equation could be composed:

$$\frac{d^2V(\tau)}{d\tau^2}-\frac{\varepsilon}{1+\varepsilon\tau}\frac{dV(\tau)}{d\tau}+\frac{\omega^2}{(1+\varepsilon\tau)^2}V(\tau)=0 \qquad (18)$$

This equation belongs to the Euler type of ordinary differential equation and its general solution is :

$$V(\tau)=(1+\varepsilon\tau)\left\{C_1\sin\left[\alpha\ln(1+\varepsilon\tau)\right]+C_2\cos\left[\alpha\ln(1+\varepsilon\tau)\right]\right\} \qquad (19)$$

where $\alpha=\sqrt{\left(\omega/\varepsilon\right)^2-1}$. From solution (19) one can conclude that amplitude of linear vibration of the linearly growing undamped rod grows proportionally to time τ .

NUMERICAL ANALYSIS

For the purposes of numerical analysis of the linearly growing rod we composed the truncated system of $N=10$ ordinary differential equations of type (17). The initial conditions correspond to deformation of the rod on the first form of the corresponding non-growing rod of the unit length:

$$z_m(0)=0,\ z_1(0)=\sin\left(\frac{\pi x}{2}\right),\ \frac{dz_m}{d\tau}(0)=0 \qquad (20)$$

Solution of equations for the first four modes is shown in Fig. 1 - 4.

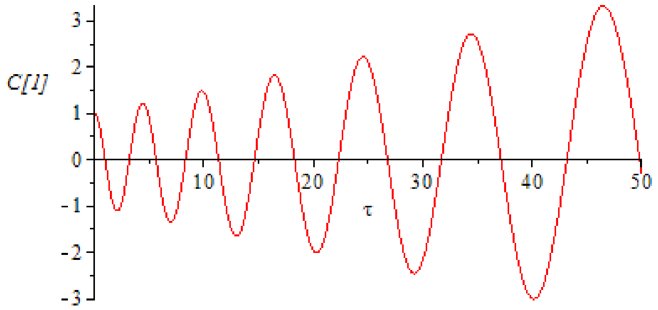

Figure 1. Resonant behavior of vibration of the classical rod at the first mode (solution of the modelling equation)

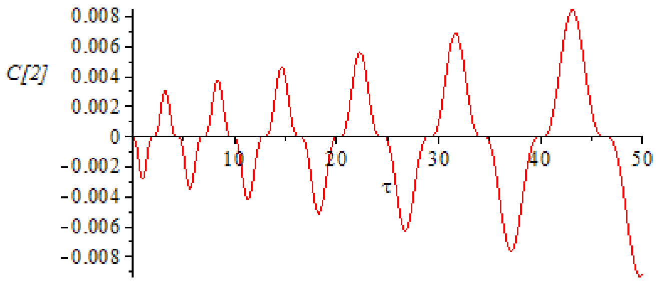

Figure 2. Resonant behavior of vibration of the classical rod at the second mode

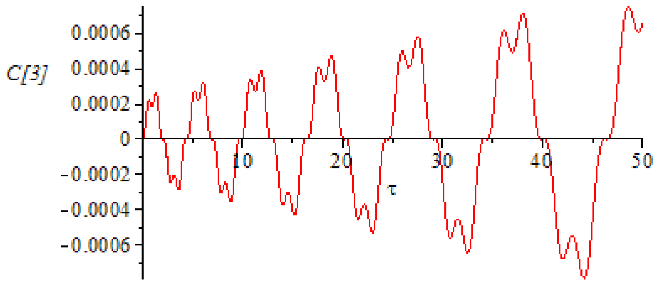

Figure 3. Resonant behavior of vibration of the classical rod at the third mode

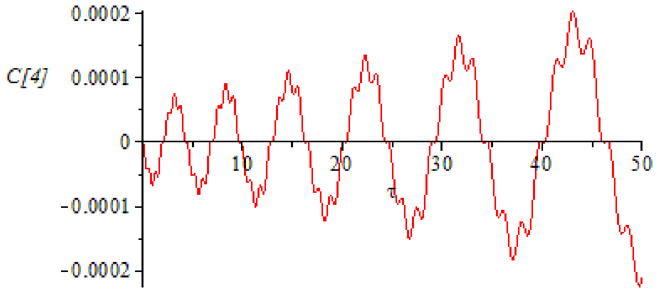

Figure 4. Resonant behavior of vibration of the classical rod
at the fourth mode

Further the similar solutions with initial conditions (20) were performed for $4 \le N < 10$. It was found that solutions for the first four modes are visually indistinguishable from solutions in Fig. 1 - 4. Moreover, the graph of solution (19) of modelling equation (18) with initial conditions $V(0)=1,\ dV/d\tau(0)=0$ is visually indistinguishable from the plot in Fig. 1. Results of numerical analysis of equations (8) and (11) of the Rayleigh-Love and Rayleigh-Bishop models obtained by the described method of transformation of the original partial differential to the systems of the ordinary differential equations demonstrate the qualitative similarities with the classical model.

Conclusions

In this article, the problem of vibration of growing rod is considered. Three different models: linear classical, Rayleigh-Love and Rayleigh-Bishop models are analyzed. It is assumed that the rod is fixed at one end and free at the other end and its length is increasing. It is demonstrated that these equations are hyperbolic and the amplitudes of vibration of the rod are growing with time. In a particular case, if the rod length is increasing proportionally to time the amplitudes are also growing proportionally to time. Further investigation demonstrates the second interesting effect: if a particular mode is excited it excites other modes and the mechanism of the modes excitation is asymmetric: the low frequency modes have higher amplitudes in comparison with the higher frequency modes.

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