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Coordinate invariance in stochastic singular optics

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Abstract

The study of optical vortices in stochastic optical fields involves various quantities, including the vortex density and topological charge density, that are defined in terms of local expectation values of the distributions of optical vortices. The complexity of these quantities often poses a formidable challenge. Here, we address this challenge with the aid of the invariance that these quantities have with respect to rotations of the coordinate axes. This property allows one to express the quantities in terms of singlets of the $SO(2)$ group that represents the coordinate rotations, resulting in expressions that are significantly simpler. We also show that the singlets can help to identify relationships among the different quantities.

Keywords: stochastic singular optics, optical vortex density, topological charge density, coordinate invariance, $SO(2)$ singlet

1. Introduction

Random optical fields, i.e. speckle [1, 2], are known to contain phase singularities [3], also called optical vortices [4]. Much work has been carried out on the statistical properties of optical vortices in random optical fields [5–14]. The study of optical vortices and other singularities in optical fields is called *singular optics* [15].

A speckle field, which is readily produced as the far-field diffraction pattern of an optical beam after passing through a ground glass plate, has statistical properties that are stationary with respect to propagation distance, up to a possible scaling. The vortices in a random optical field are evenly distributed, with a density given by the second derivative of the peak of its autocorrelation function [5]. Neighbouring vortices tend to have opposite topological charges [10], leading to a topological charge density that is on average zero [13, 16].

Nature also provides situations where the statistical properties of optical fields vary with propagation distance. One example is found in the scintillation of an optical beam propagating through a random medium such as turbulence [17]. In a certain sense one can view the process whereby random phase modulations turn an optical field into fully developed speckle as another example of such a case [18]. Other examples include cases where the initial

optical field contains lateral correlations in its angular spectrum (far-field distribution) [19–21]. All such optical fields are here referred to as *stochastic optical fields* to distinguish them from random optical fields, which have stationary statistical properties. The study of the statistical properties of optical vortices in stochastic optical fields is referred to as *stochastic singular optics*.

One of the aims of stochastic singular optics is to understand the relationships among the various local quantities, such as the vortex density, topological charge density, phase gradient, etc, that can be computed from stochastic optical fields, using statistical optics methods. However, such an investigation is hampered by the complexity of the expressions that one often encounters when computing these local quantities. In this paper we address this challenge by exploiting the invariance that all such quantities have with respect to rotations of the transverse coordinate axes, which can be represented as an $SO(2)$ Lie group. The coordinate invariance allows one to define $SO(2)$ singlets, in terms of which the expressions for the relevant quantities become significantly simpler. The simpler expressions allow one to find relationships among the different quantities, which would otherwise be lost in the complexity. The analysis in this paper is restricted to monochromatic (temporally coherent), paraxial optical fields.

The paper is organized as follows. In section 2 we briefly review the calculation procedure for the local quantities and show that this process can be divided into two parts. The SO(2) coordinate invariance is introduced and discussed in section 3. In section 4 we demonstrate the benefit of using the SO(2) singlets in these expressions by showing how they reveal relationships among the local quantities. Some conclusions are provided in section 5, and for the reader's benefit we list some of the SO(2) singlets and SO(2) doublets in appendices A and B, respectively.

2. The general procedure to calculate the expectation values of local quantities

One can separate the calculation of the expectation values of local quantities into two parts. One part is to obtain an expression for the local quantity in terms of local field correlation functions. This part needs to be carried out only once and the result is valid for all stochastic optical fields. The other part is to compute the local field correlation functions for a particular stochastic optical field, which can then be substituted into the general expression for a local quantity to obtain the local quantity for that stochastic optical field. Here, we first discuss the local two-point correlation functions and then provide the general procedure to compute the expressions for the expectation values of local quantities in terms of these local two-point correlation functions.

2.1. General local two-point field correlation functions

The simplest and best known example of a local quantity is the average intensity, given by the expectation value of the modulus squared of the optical field, $I(x, y, z) = \langle g\bar{g} \rangle$. As such it is a local two-point field correlation function. One can use the non-local two-point correlation function, which is also called the mutual coherence function [22, 23],

$$\Gamma_{\text{full}}(\mathbf{x}_1, t_1, \mathbf{x}_2, t_2) = \langle g(\mathbf{x}_1, t_1)\bar{g}(\mathbf{x}_2, t_2) \rangle \quad (1)$$

to compute such local correlation functions. In (1), $\mathbf{x} = \{x, y, z\}$, $g(\cdot)$ is the complex scalar optical field, $\langle \cdot \rangle$ denotes the expectation value of a quantity and \bar{g} indicates the complex conjugate of g . We assume that the optical field is monochromatic. Hence, the time dependence is unimportant and can be dropped. We also consider the correlation functions on a plane with the same z -coordinate for both points. The mutual coherence function then becomes

$$\Gamma(x_1, y_1, x_2, y_2, z) = \langle g(x_1, y_1, z)\bar{g}(x_2, y_2, z) \rangle. \quad (2)$$

The intensity is obtained from this mutual coherence function by setting $x_2 = x_1 = x$ and $y_2 = y_1 = y$. Hence, $I(x, y, z) = \Gamma(x, y, x, y, z)$.

We are also interested in local correlation functions between the optical field and derivatives of the optical field. For instance,

$$\langle g\bar{g}_x \rangle = \langle g(x, y, z)\partial_x\bar{g}(x, y, z) \rangle. \quad (3)$$

Since the derivative and the expectation value both represent linear operations, they can be interchanged, $\partial_x\langle g \rangle = \langle \partial_x g \rangle$.

Hence, one can use the mutual coherence function in (2) as a generating function for all such local field correlation functions, provided that the correlation is made non-locally prior to the differential operation and only made local afterwards. For example,

$$\begin{aligned} \langle g\bar{g}_x \rangle &= [\partial_x \langle g(u, v, z)\bar{g}(x, y, z) \rangle]_{u=x, v=y} \\ &= [\partial_x \Gamma(u, v, x, y, z)]_{u=x, v=y}. \end{aligned} \quad (4)$$

2.2. The general procedure to calculate expectation values of local quantities

The expectation value of a quantity W , which is a function of a set of random variables $\mathbf{q} = \{q_n\}$ for $n = 1, \dots, N$, is given by

$$\langle W \rangle = \int W(\mathbf{q})F_{\mathbf{q}}(\mathbf{q}) d^N q, \quad (5)$$

where $F_{\mathbf{q}}(\mathbf{q})$ is the joint probability density function for the q_n s.

In the present case, the q_n s are auxiliary variables replacing the real and imaginary parts of the optical field and its derivatives. For the case where we restrict ourselves to correlation functions involving up to first derivatives of the optical field, there are six q_n s, so that $N = 6$ in (5). The optical field and its first derivatives can be expressed as a vector

$$\mathbf{G} = [g(\mathbf{x}), \partial_x g(\mathbf{x}), \partial_y g(\mathbf{x})]^T, \quad (6)$$

giving nine local two-point field correlation functions, which can be expressed in terms of the covariance matrix

$$\langle \mathbf{G}\mathbf{G}^\dagger \rangle = M_1 = \begin{bmatrix} \langle g\bar{g} \rangle & \langle g\bar{g}_x \rangle & \langle g\bar{g}_y \rangle \\ \langle g_x\bar{g} \rangle & \langle g_x\bar{g}_x \rangle & \langle g_x\bar{g}_y \rangle \\ \langle g_y\bar{g} \rangle & \langle g_y\bar{g}_x \rangle & \langle g_y\bar{g}_y \rangle \end{bmatrix}. \quad (7)$$

Since the covariance matrix is Hermitian, it only contains nine real-valued degrees of freedom. The joint probability density function is then given by [5, 14, 20]

$$F_{\mathbf{q}}(\mathbf{q}) = \frac{\exp(-\mathbf{Q}^\dagger M_1^{-1} \mathbf{Q})}{\pi^3 \det(M_1)}, \quad (8)$$

where the q_n s are combined into a complex vector

$$\mathbf{Q} = [q_1 + iq_2, q_3 + iq_4, q_5 + iq_6]^T. \quad (9)$$

The expression of $W(\mathbf{q})$ depends on the specific quantity under investigation. For the vortex density, it is

$$W_V(\mathbf{q}) = \delta(q_1)\delta(q_2)|q_3q_6 - q_4q_5|, \quad (10)$$

for the topological charge density, it is given by

$$W_T(\mathbf{q}) = \delta(q_1)\delta(q_2)(q_3q_6 - q_4q_5) \quad (11)$$

and for the local phase gradient it reads

$$W_F(\mathbf{q}) = \frac{(q_3q_2 - q_4q_1)\hat{x} + (q_5q_2 - q_6q_1)\hat{y}}{q_1^2 + q_2^2}. \quad (12)$$

The expressions that are obtained by evaluating the q_n -integrals for these quantities are provided in the discussion below.

3. Orthogonal SO(2) singlets

3.1. Coordinate invariance

The expressions for the expectation values of local quantities, such as the topological charge density and the vortex density, contain multivariate polynomials in terms of the field correlation functions. These multivariate polynomials consist of sums of terms, each with a product of field correlation functions. Naturally, the different terms in such an expansion carry the same dimensions, which means that the number of fields and the number of derivatives are the same in all terms.

Consider, for example, the topological charge density, which is given by

$$T = \frac{i}{2\pi \langle g\bar{g} \rangle^2} (\langle g\bar{g} \rangle \langle g_x\bar{g}_y \rangle - \langle g\bar{g} \rangle \langle g_y\bar{g}_x \rangle + \langle g\bar{g}_x \rangle \langle g_y\bar{g} \rangle - \langle g\bar{g}_y \rangle \langle g_x\bar{g} \rangle). \quad (13)$$

The numerator is a multivariate polynomial, in which the different correlation functions represent different ‘variables’—i.e. different degrees of freedom. Each of the four terms in the polynomial consists of the product of two field correlation functions (four optical fields in total) and each term contains a total of two derivatives.

The choice of coordinate system is arbitrary. In other words, one may decide to rotate the x - and y -axes of the coordinate system by an arbitrary angle. The expression of the expectation value of the local quantity should not be affected by such a coordinate transformation. This implies that the multivariate polynomials in the expressions of local quantities are always invariant with respect to such coordinate transformations.

Rotations in the (x, y) -plane can be represented by the elements of the SO(2) Lie group¹ and the invariant multivariate polynomials are singlets of this SO(2) Lie group.

3.2. Transformations of field correlation functions

Under the SO(2) Lie group of rotations in the (x, y) -plane, the covariance matrix in (7) transforms as follows:

$$M_1 \rightarrow OM_1O^{-1}, \quad (14)$$

where $O^{-1} = O^T$ and

$$O = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix} \quad (15)$$

is a reducible element of SO(2), with α being the rotation angle.

Since (15) is reducible, the transformations that it imposes on the individual field correlation functions can be divided into different groups that transform differently. One

can extract the transformation rules for the individual field correlation functions in these different groups from (7), (14) and (15). The first group has just one element, the intensity, which does not transform at all $\langle g\bar{g} \rangle \rightarrow \langle g\bar{g} \rangle$. The next group transforms as a doublet under SO(2),

$$\begin{aligned} \langle g_x\bar{g} \rangle &\rightarrow \langle g_x\bar{g} \rangle \cos(\alpha) - \langle g_y\bar{g} \rangle \sin(\alpha) \\ \langle g_y\bar{g} \rangle &\rightarrow \langle g_x\bar{g} \rangle \sin(\alpha) + \langle g_y\bar{g} \rangle \cos(\alpha). \end{aligned} \quad (16)$$

The same is true for the group consisting of $\langle g\bar{g}_x \rangle$ and $\langle g\bar{g}_y \rangle$. The last group, consisting of $\langle g_x\bar{g}_x \rangle$, $\langle g_y\bar{g}_y \rangle$, $\langle g_y\bar{g}_x \rangle$ and $\langle g_x\bar{g}_y \rangle$, and other groups containing higher derivatives, generally transform as tensor products of doublets.

3.3. Scalar singlets

To be invariant under the coordinate rotation, the multivariate polynomials must consist of combinations of field correlation functions, such that their transformations cancel out any additional terms that are formed individually. Consider, for example, the combination $\langle g\bar{g}_x \rangle \langle g_y\bar{g} \rangle - \langle g\bar{g}_y \rangle \langle g_x\bar{g} \rangle$, which appears in the numerator of the topological charge density in (13). When one applies the transformations in (16), etc to this combination, it remains unchanged. The same is true when one applies the appropriate transformations to the other two terms, $\langle g_x\bar{g}_y \rangle - \langle g_y\bar{g}_x \rangle$. Since $\langle g\bar{g} \rangle$ does not transform, the entire expression for the topological charge density in (13) is invariant with respect to a rotation of the coordinate axes, as expected.

This invariance is a useful property that can help to simplify the expressions for the expectation values of local quantities. In terms of the field correlation functions, these expressions can be rather complicated, and when differential operations are performed on them the results become even more complicated. It is often challenging to make sense of such a result. With the aid of SO(2) singlets, one can simplify the expressions and thus be able to recognize the expressions of particular local quantities.

For this purpose, it is beneficial to identify all the SO(2) singlets that can be formed as combinations of the field correlation functions up to an appropriate level. The SO(2) singlets that consist of the field correlation functions in (7) are denoted as τ_n s. A list of SO(2) singlets is provided in appendix A, with the simplest one being the intensity, $\tau_0 = \langle g\bar{g} \rangle$. The topological charge density in (13) can now be expressed as

$$T = \frac{\tau_0\tau_2 - \tau_6}{2\pi\tau_0^2}, \quad (17)$$

giving a much simpler expression.

3.4. Vector singlets

The singlets in the topological charge density are all scalars. One can also have singlets that are vectors on the (x, y) -plane. Such vectorial quantities include the local phase gradient, as

¹ Since the SO(2) Lie group is isomorphic to the U(1) Lie group, one can also use the latter. However, for our purposes here it is more convenient to consider the transformation in terms of two-dimensional real-valued matrices instead of one-dimensional complex-valued factors.

well as the gradient of any scalar quantity. They transform as SO(2) singlets, because the unit vectors transform as follows:

$$\begin{aligned}\hat{x} &\rightarrow \hat{x} \cos(\alpha) - \hat{y} \sin(\alpha) \\ \hat{y} &\rightarrow \hat{x} \sin(\alpha) + \hat{y} \cos(\alpha).\end{aligned}\quad (18)$$

To define the vectorial quantities in terms of SO(2) singlets, one also needs to identify all the SO(2) singlets of such vectorial spaces—i.e. spaces where the terms contain a factor of \hat{x} or \hat{y} .

The local phase gradient can be expressed in terms of SO(2) singlets as

$$\mathbf{F} = \frac{\mathbf{v}_2}{2\tau_0}, \quad (19)$$

where

$$\mathbf{v}_2 = -i[(\langle g_x \bar{g} \rangle - \langle g \bar{g}_x \rangle)\hat{x} + (\langle g_y \bar{g} \rangle - \langle g \bar{g}_y \rangle)\hat{y}] \quad (20)$$

is a vectorial SO(2) singlet. Another vectorial singlet is

$$\mathbf{v}_1 = (\langle g_x \bar{g} \rangle + \langle g \bar{g}_x \rangle)\hat{x} + (\langle g_y \bar{g} \rangle + \langle g \bar{g}_y \rangle)\hat{y}, \quad (21)$$

which we will encounter again later.

3.5. Inner product spaces

The reason why the numerator of the topological charge density in (17) is expressed in terms of three invariant combinations instead of just one is because the terms of one of these combinations cannot transform into the terms of the other combinations. Consider, for instance, τ_2 . Its terms can only transform into $\langle g_x \bar{g}_y \rangle$, $\langle g_y \bar{g}_x \rangle$, $\langle g_x \bar{g}_x \rangle$ or $\langle g_y \bar{g}_y \rangle$. These four terms define a four-dimensional space of possible combinations. In the case of τ_6 the different possible terms are $\langle g_x \bar{g} \rangle \langle g \bar{g}_y \rangle$, $\langle g_y \bar{g} \rangle \langle g \bar{g}_x \rangle$, $\langle g_x \bar{g} \rangle \langle g \bar{g}_x \rangle$ and $\langle g_y \bar{g} \rangle \langle g \bar{g}_y \rangle$. This is a different four-dimensional space. Only a proper subset of each of these four-dimensional spaces contains combinations that are SO(2) singlets. From the group theory perspective, these four-dimensional spaces consist of disjoint subspaces that transform under different irreducible representations of the SO(2) Lie group, with the singlet being the simplest irreducible representation. We are only interested in the part of the space that transforms as singlets. However, knowledge of the doublets is useful in computing all the singlets.

In general, the space defined by the terms of a particular form could have any number of dimensions and the same is true for its invariant subspace, which contains all the singlets of that space. If the invariant subspace has more than one dimension, which is usually the case, then there are various different ways to define singlets for that invariant subspace—all linear combinations of singlets are again singlets. Therefore, the definitions of these singlets are not unique. To avoid this ambiguity we define an inner product for these invariant subspaces and then define an orthogonal basis for it. The orthogonal basis elements are referred to as *orthogonal singlets*. The larger spaces can have some basis elements that are constructed out of products of the orthogonal singlets of smaller spaces. Such elements are said to be *reducible*—not to be confused with the notion of a reducible

representation in group theory. The *irreducible* singlets are listed in appendix A. With the exception of τ_7 , they are all orthogonal singlets.

3.6. Complex conjugation

Although the field correlation functions are in general complex-valued, the expectation values of local quantities are always real-valued. While one can form complex-valued combinations that are SO(2) singlets, all the singlets in appendix A are defined to be real-valued. That is the reason for the ‘i’s in the definitions in appendix A. Without the ‘i’s, the singlets in appendix A are eigenfunctions of complex conjugation with eigenvalues ± 1 . In other words, they are either purely real or purely imaginary.

3.7. Mirror transformation

Another property of the orthogonal singlets is that they are eigenfunctions of the mirror transformation on the (x, y) -plane. One can define such a mirror transformation by interchanging the x - and y -derivatives, which implies a mirror transformation with respect to the line on which $x = y$. Due to the invariance with respect to the rotation of the coordinate axes, this mirror transformation is equivalent to all mirror transformations with respect to any line through the origin of the (x, y) -plane. The eigenvalues are either 1 or -1 , which respectively indicate a symmetric or an anti-symmetric function.

The anti-symmetric orthogonal singlets are closely related to the topological charge. For instance, both τ_2 and τ_6 are anti-symmetric, while the intensity τ_0 is symmetric. This implies that the topological charge density given in (17) is anti-symmetric as a whole.

3.8. Sets of field correlation functions

The correlation functions in (7) can be divided into three sets, depending on their number of derivatives. These sets are as follows.

- *Set 0*. Contains one element $\langle g \bar{g} \rangle$.
- *Set 1*. Contains four elements of the form $\langle g \bar{g} \square \rangle$ and $\langle g \square \bar{g} \rangle$,
 $\{\langle g_x \bar{g} \rangle, \langle g_y \bar{g} \rangle, \langle g \bar{g}_x \rangle, \langle g \bar{g}_y \rangle\}$.
- *Set 2*. Contains four elements of the form $\langle g \square \bar{g} \square \rangle$,
 $\{\langle g_x \bar{g}_x \rangle, \langle g_y \bar{g}_y \rangle, \langle g_x \bar{g}_y \rangle, \langle g_y \bar{g}_x \rangle\}$.

Set 0 is a trivial set with only one element, the intensity, which is already an SO(2) singlet. A general invariant polynomial can be expressed as a polynomial in $\langle g \bar{g} \rangle$, with coefficients that are themselves SO(2) singlets. As a result, we do not need to consider multivariate polynomials that contain $\langle g \bar{g} \rangle$.

The elements of each of these sets can be transformed into each other via coordinate rotation, mirror transformation or complex conjugation, but never into elements of other sets. These sets are used to distinguish the different spaces for the invariant polynomials.

3.9. Computing orthogonal singlets from doublets

We are interested in the singlets that can be formed as linear combinations of terms consisting of specific products of the elements from the different sets in section 3.8. The SO(2) singlets are formed by composing them out of SO(2) doublets and the doublets are formed by building them up from the simplest SO(2) doublets, which are provided in appendix B. All irreducible representations of the SO(2) coordinate rotations are doublets, with the singlets being the elements of doublets of spin zero. The spin is defined below. The simplest doublets are those with components consisting of linear combinations of the elements of one of the sets in section 3.8.

The doublets transform as

$$D \rightarrow D' = O_n(\alpha)D, \quad (22)$$

where

$$O_n(\alpha) = \begin{bmatrix} \cos(n\alpha) & -\sin(n\alpha) \\ \sin(n\alpha) & \cos(n\alpha) \end{bmatrix} \quad (23)$$

is a 2×2 rotation matrix with α being the coordinate rotation angle and n being an integer, which we refer to as the *spin* of the doublet. The doublets, $D = [a, b]^T$ and $D' = [a', b']^T$, have the same spin. A doublet's two components a and b have the same eigenvalue under complex conjugation—i.e. they are either both real or both imaginary. (One can also define these doublets to be complex-valued, in which case some of the doublets would be complex conjugates of each other.) Under the mirror transformation a is always symmetric and b is always anti-symmetric.

Doublets can be combined to form new doublets, using the 2×2 identity matrix

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (24)$$

and the Pauli spin matrices

$$\begin{aligned} \sigma_1 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \sigma_2 &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\ \sigma_3 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \end{aligned} \quad (25)$$

Given the two doublets D_1 and D_2 , with spins m and n , respectively, one can form the following four quantities:

$$\delta_0 = D_1^T \sigma_0 D_2 = a_1 a_2 + b_1 b_2 \quad (26)$$

$$\delta_1 = D_1^T \sigma_1 D_2 = a_1 b_2 + b_1 a_2 \quad (27)$$

$$\delta_2 = i D_1^T \sigma_2 D_2 = a_1 b_2 - b_1 a_2 \quad (28)$$

$$\delta_3 = D_1^T \sigma_3 D_2 = a_1 a_2 - b_1 b_2, \quad (29)$$

which are combined into two new doublets $D_3 = [\delta_0, \delta_2]^T$ and $D_4 = [\delta_3, \delta_1]^T$, with spins $(n - m)$ and $(n + m)$, respectively. If $m = n$, the elements of D_3 are singlets and, if $D_1 = D_2$, the only singlet in D_3 is δ_0 , with $\delta_2 = 0$.

4. Relationships among local quantities

To demonstrate the benefit of using orthogonal singlets, we discuss here a number of relationships that exist among the local quantities. We also discuss the derivatives of the field correlation functions, which allow one to find differential relationships among the quantities.

4.1. Vortex density and related quantities

The expression of the vortex density in terms of orthogonal singlets is given by

$$V(\mathbf{x}) = \frac{2\tau_0(\tau_0\tau_7 - \epsilon_1) + (\tau_0\tau_2 - \tau_6)^2}{2\pi\tau_0^2\sqrt{4\tau_0(\tau_0\tau_7 - \epsilon_1) + (\tau_0\tau_2 - \tau_6)^2}}, \quad (30)$$

where

$$\begin{aligned} \epsilon_1 &= \frac{1}{2}(\tau_1\tau_5 - \tau_2\tau_6 - \tau_9) \\ &= \langle g_x \bar{g}_x \rangle \langle g_y \bar{g}_y \rangle \langle g \bar{g}_y \rangle + \langle g_y \bar{g}_y \rangle \langle g_x \bar{g}_x \rangle \langle g \bar{g}_x \rangle \\ &\quad - \langle g_x \bar{g}_y \rangle \langle g_y \bar{g}_x \rangle \langle g \bar{g}_x \rangle - \langle g_y \bar{g}_x \rangle \langle g_x \bar{g}_y \rangle \langle g \bar{g}_y \rangle, \end{aligned} \quad (31)$$

$\tau_0 = \langle g \bar{g} \rangle$ and the other τ_n s are provided in appendix A. The combinations of SO(2) singlets that appear in (30) also appear in other local quantities. For instance, comparing (17) and (30), one notices that the same combination $(\tau_0\tau_2 - \tau_6)$ appears in both. Moreover, the determinant of the covariance matrix

$$\det(M_1) = \tau_0\tau_7 - \epsilon_1 \quad (32)$$

appears in the expression for the vortex density. The vortex density can therefore be simplified to

$$V = \frac{Q + 2T^2}{2\sqrt{Q + T^2}} = \frac{1}{2}\sqrt{Q + T^2} + \frac{T^2}{2\sqrt{Q + T^2}}, \quad (33)$$

where T is the topological charge density (17) and for convenience we combined the determinant of the covariance matrix and the intensity τ_0 into

$$Q = \frac{\det(M_1)}{\pi^2\tau_0^3}. \quad (34)$$

Conversely, the determinant of the covariance matrix can be given purely in terms of the intensity τ_0 , the topological charge density T and the vortex density V ,

$$\det(M_1) = 2\pi^2\tau_0^3 \left[R + \sqrt{R^2 + T^2R} \right], \quad (35)$$

where $R = V^2 - T^2$. The determinant of the covariance matrix and the vortex density are positive definite quantities. The topological charge density can be positive or negative, but its magnitude cannot exceed the vortex density, $V \geq |T|$. For $T = 0$, the determinant simplifies to $\det(M_1) = 4\pi^2\tau_0^3V^2$.

The positive and negative vortex distributions n_p and n_n are defined in terms of the sums and differences of the vortex density V and the topological charge density T . Using (33), one finds the following expression for these quantities:

$$\left. \begin{aligned} n_p \\ n_n \end{aligned} \right\} = \frac{1}{2}(V \pm T) = \frac{(\sqrt{Q + T^2} \pm T)^2}{4\sqrt{Q + T^2}}. \quad (36)$$

4.2. Transverse derivatives of field correlation functions

Some of the relationships among the expectation values of local quantities involve transverse derivatives, i.e. derivatives with respect to x and y . Transverse derivatives also appear in the dynamical equations [19–21].

To see the effect of such transverse derivatives on local field correlation functions, we apply them to the example in (4),

$$\begin{aligned} \partial_y \langle g \bar{g}_x \rangle &= \partial_y \{ [\partial_x \langle g(u, v, z) \bar{g}(x, y, z) \rangle]_{u=x, v=y} \} \\ &= [\partial_v \partial_x \Gamma(u, v, x, y, z) \\ &\quad + \partial_y \partial_x \Gamma(u, v, x, y, z)]_{u=x, v=y} \\ &= \langle g_y \bar{g}_x \rangle + \langle g \bar{g}_{xy} \rangle. \end{aligned} \quad (37)$$

The derivatives that we refer to here need to be distinguished from those that are applied to the mutual coherence function to generate the original set of field correlation functions. In contrast, these derivatives are applied to the field correlation functions after the coordinates of the two points have been set equal to each other. Therefore, the new derivatives operate on both sets of coordinates in the mutual coherence function. As a result the derivative of a field correlation function always gives a sum of two other field correlation functions, each with one more derivative than the original field correlation function. The second field correlation function in (37) contains the second derivative of the optical field. Hence, the transverse derivatives of field correlation functions generate new field correlation functions beyond those in the covariance matrix M_1 in (7).

The general rule for the transverse derivatives of field correlation functions is

$$\partial_r \langle g_p \bar{g}_q \rangle = \langle g_{rp} \bar{g}_q \rangle + \langle g_p \bar{g}_{rq} \rangle, \quad (38)$$

where r is either x or y and p and q can be any combination of xs and ys .

If the derivatives are applied as differential operators $\nabla \equiv \hat{x} \partial_x + \hat{y} \partial_y$, which are defined on the two-dimensional transverse plane, the result is again a singlet of the SO(2) Lie group of coordinate rotations. As examples, the gradient of the intensity becomes

$$\begin{aligned} \nabla \tau_0 &= \partial_x \langle g \bar{g} \rangle \hat{x} + \partial_y \langle g \bar{g} \rangle \hat{y} \\ &= (\langle g_x \bar{g} \rangle + \langle g \bar{g}_x \rangle) \hat{x} + (\langle g_y \bar{g} \rangle + \langle g \bar{g}_y \rangle) \hat{y} \\ &= \mathbf{v}_1, \end{aligned} \quad (39)$$

where we used the definition of \mathbf{v}_1 in (21). An example of a divergence is

$$\begin{aligned} \nabla \cdot \mathbf{v}_2 &= -i[\partial_x (\langle g_x \bar{g} \rangle - \langle g \bar{g}_x \rangle) + \partial_y (\langle g_y \bar{g} \rangle - \langle g \bar{g}_y \rangle)] \\ &= -i(\langle g_{xx} \bar{g} \rangle - \langle g \bar{g}_{xx} \rangle + \langle g_{yy} \bar{g} \rangle - \langle g \bar{g}_{yy} \rangle), \end{aligned} \quad (40)$$

which is also a singlet. Finally, we consider the curl

$$\begin{aligned} \nabla \times \mathbf{v}_2 &= -i[\partial_x (\langle g_y \bar{g} \rangle - \langle g \bar{g}_y \rangle) - \partial_y (\langle g_x \bar{g} \rangle - \langle g \bar{g}_x \rangle)] \hat{z} \\ &= i2(\langle g_x \bar{g}_y \rangle - \langle g_y \bar{g}_x \rangle) \hat{z} = 2\tau_2 \hat{z}. \end{aligned} \quad (41)$$

We see that in these cases the differential operators turn orthogonal singlets into other orthogonal singlets. In general, the result of such a differential operation may be a linear combination of orthogonal singlets.

4.3. The curl of the local phase gradient

The fact that the phase function of an optical field can contain a phase singularity is an indication that phase functions are special kinds of functions. While functions usually have unbounded, simply connected ranges, the range of a phase function is compact and not simply connected. In fact, its range has the topology of a circle. If one considers the mapping of a closed contour in the domain of the phase function to this circle for the phase values, one can find situations where this mapping wraps around the circle. The continuity of the function now implies that if the closed contour shrinks continuously to a point in the domain, the mapping of the contour onto the circle should also shrink to a point, but because it is wrapped around the circle this is not possible. Hence, the initial contour in the domain must enclose a singularity. If the contour passes over this singular point in the domain, the mapping jumps from one side of the circle to the other.

The presence of such phase singularities also implies that derivatives do not commute. For functions with unbounded, simply connected ranges we have that $\nabla \times \nabla f(\mathbf{x}) = 0$, but for phase functions [24, 25]

$$\nabla \times \nabla \theta(\mathbf{x}) = 2\pi \sum_n v_n \delta(\mathbf{x} - \mathbf{x}_n) \hat{z}, \quad (42)$$

where the \mathbf{x}_n s represent the locations of the phase singularities and the v_n s are the topological charges (signed integers) associated with these respective phase singularities.

For a stochastic optical field, the expression in (42) translates into a relationship between the local phase gradient and the topological charge density,

$$\nabla \times \mathbf{F}(\mathbf{x}) = 2\pi T(\mathbf{x}) \hat{z}. \quad (43)$$

One can verify (43) by using the expressions for T and \mathbf{F} in terms of orthogonal singlets, (17) and (19), respectively. Applying the curl to (19), we obtain

$$\begin{aligned} \nabla \times \left(\frac{\mathbf{v}_2}{2\tau_0} \right) &= \frac{\nabla \times \mathbf{v}_2}{2\tau_0} - \frac{\nabla \tau_0 \times \mathbf{v}_2}{2\tau_0^2} \\ &= \frac{(\tau_0 \tau_2 - \tau_6) \hat{z}}{\tau_0^2} = 2\pi T(\mathbf{x}) \hat{z}, \end{aligned} \quad (44)$$

where we used (39) and (41), together with the identity

$$\mathbf{v}_1 \times \mathbf{v}_2 = 2\tau_6 \hat{z}, \quad (45)$$

which can be confirmed from (20) and (21) and the expressions for τ_6 in appendix A. Hence, using orthogonal singlets, we confirmed that the relationship in (43) is valid for stochastic optical fields.

4.4. Longitudinal derivatives of field correlation functions

To describe the evolution of local quantities in stochastic optical fields as a function of the propagation distance one needs to consider the effect of longitudinal derivatives on the field correlation functions with respect to the propagation distance. Under the paraxial approximation the longitudinal

derivative of an optical field is related to the second order transverse derivatives via the paraxial wave equation

$$\begin{aligned}\partial_z g(x, y, z) &= \frac{-i}{2k} \left[\partial_x^2 g(x, y, z) + \partial_y^2 g(x, y, z) \right] \\ \partial_z \bar{g}(x, y, z) &= \frac{i}{2k} \left[\partial_x^2 \bar{g}(x, y, z) + \partial_y^2 \bar{g}(x, y, z) \right],\end{aligned}\quad (46)$$

where $k = 2\pi/\lambda$ is the wavenumber. Applying the longitudinal derivative to the example in (4) and using the paraxial wave equation (46), we obtain

$$\begin{aligned}\partial_z \langle g \bar{g}_x \rangle &= \left[\partial_z \partial_x \langle g(u, v, z) \bar{g}(x, y, z) \rangle \right]_{u=x, v=y} \\ &= \left(\left[\partial_z g(x, y, z) \right] \left[\partial_x \bar{g}(x, y, z) \right] \right. \\ &\quad \left. + \langle g(x, y, z) \left[\partial_x \partial_z \bar{g}(x, y, z) \right] \right) \\ &= \frac{i}{2k} (\langle g \bar{g}_{xxx} \rangle + \langle g \bar{g}_{xyy} \rangle \\ &\quad - \langle g_{xx} \bar{g}_x \rangle - \langle g_{yy} \bar{g}_x \rangle).\end{aligned}\quad (47)$$

The result contains four new field correlation functions, each with two more derivatives than the original field correlation function.

The general rule for longitudinal derivatives of field correlation functions is

$$\begin{aligned}\partial_z \langle g_p \bar{g}_q \rangle &= \frac{-i}{2k} (\langle g_{xyp} \bar{g}_q \rangle - \langle g_p \bar{g}_{xxq} \rangle \\ &\quad + \langle g_{pyy} \bar{g}_q \rangle - \langle g_p \bar{g}_{qyy} \rangle),\end{aligned}\quad (48)$$

where p and q can be any combination of x s and y s. As a rule we always put the x s before the y s in the subscripts denoting derivatives.

4.5. Intensity transport

Applying the longitudinal derivative to the intensity $\tau_0 = \langle g \bar{g} \rangle$, one obtains

$$\begin{aligned}\partial_z \tau_0 &= \frac{-i}{2k} (\langle g_{xx} \bar{g} \rangle - \langle g \bar{g}_{xx} \rangle + \langle g_{yy} \bar{g} \rangle - \langle g \bar{g}_{yy} \rangle) \\ &= \frac{\nabla \cdot \mathbf{v}_2}{2k},\end{aligned}\quad (49)$$

where we used (40) to obtain the final expression. Since \mathbf{v}_2 appears in the local phase gradient (19), let us consider the divergence of the local phase gradient. We obtain

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \nabla \cdot \left(\frac{\mathbf{v}_2}{2\tau_0} \right) = \frac{\nabla \cdot \mathbf{v}_2}{2\tau_0} - \frac{\nabla \tau_0 \cdot \mathbf{v}_2}{2\tau_0^2} \\ &= \frac{k \partial_z \tau_0}{\tau_0} - \frac{\nabla \tau_0 \cdot \mathbf{F}}{\tau_0},\end{aligned}\quad (50)$$

where we substituted in (49). Hence,

$$k \partial_z \tau_0 = \tau_0 \nabla \cdot \mathbf{F} + (\nabla \tau_0) \cdot \mathbf{F} = \nabla \cdot (\tau_0 \mathbf{F}).\quad (51)$$

The expression in (51) is a dynamical equation that relates the intensity τ_0 and the local phase gradient \mathbf{F} . It is a known result [26] for deterministic optical fields, often referred to as the *intensity transport equation*. The use of orthogonal singlets made it possible to show that it also applies for stochastic optical fields.

5. Conclusions

Although the local expectation values of quantities that one can compute from stochastic optical fields often produce formidably complex expressions, one can mitigate this complexity by exploiting the invariance that these quantities have with respect to coordinate rotations. These rotations form an SO(2) group. All the quantities of interest can therefore be expressed in terms of singlets of this SO(2) group. Here we develop a formalism to express the relevant quantities in terms of these singlets. This formalism includes the definition the orthogonal singlets and the method to obtain the expressions for these orthogonal singlets. We also provide a list of the most common singlets. It is shown that the use of these orthogonal singlets allows one to find relationships among the different quantities. These relationships include pure algebraic relationships, differential relationships and dynamical equations. Examples are provided for each of them.

Here, we only consider field correlation functions with up to one derivative of the optical fields. However, orthogonal singlets can also be defined for correlation functions that contain higher derivatives. Such higher derivative correlation functions are required when one considers derivatives of local quantities, as required in dynamical equations and also for local quantities that contain higher derivative correlation functions, such as the distributions of the Poincaré–Hopf indices [27] and the probability density for annihilation and creation events. The calculations of these quantities are in general extremely complex. An indication of this complexity is the fact that, while the determinant of the covariance matrix with up to first derivatives of the optical field has six terms, the determinant of the covariance matrix with up the second derivatives of the optical field has 720 terms.

Appendix A. List of orthogonal singlets

Here we list the orthogonal bases for the invariant subspaces that can be formed with up to four transverse derivatives per term, but with at most one derivative per optical field. These orthogonal bases can consist of reducible and irreducible singlets. Reducible singlets are those that can be written in terms of products of singlets from simpler spaces. The expressions of the irreducible singlets are provided. While the simplest singlets are expressed directly in terms of the correlation functions, the more complicated singlets are expressed in terms of the ξ_n s that are defined in appendix B.

The orthogonal singlets are grouped according to the spaces that they belong to. Each space is defined in terms of the number of times that an element of a particular set from section 3.8 appears in the terms for that space. The spaces are denoted by two integers. The first (second) integer denotes the number of factors from set 1 (set 2). The orthogonal singlets, which we denote by τ_s , are not normalized and are thus not orthonormal.

- Basis 00: $\{\tau_0\}$

$$\tau_0 = \langle g \bar{g} \rangle.$$

- Basis 01: $\{\tau_1, \tau_2\}$

$$\begin{aligned}\tau_1 &= \langle g_x \bar{g}_x \rangle + \langle g_y \bar{g}_y \rangle \\ \tau_2 &= i(\langle g_x \bar{g}_y \rangle - \langle g_y \bar{g}_x \rangle).\end{aligned}$$

- Basis 20: $\{\tau_3, \tau_4, \tau_5, \tau_6\}$

$$\begin{aligned}\tau_3 &= \langle g_x \bar{g} \rangle^2 + \langle g_y \bar{g} \rangle^2 + \langle g \bar{g}_x \rangle^2 + \langle g \bar{g}_y \rangle^2 \\ \tau_4 &= i \left(\langle g_x \bar{g} \rangle^2 + \langle g_y \bar{g} \rangle^2 - \langle g \bar{g}_x \rangle^2 - \langle g \bar{g}_y \rangle^2 \right) \\ \tau_5 &= \langle g_x \bar{g} \rangle \langle g \bar{g}_x \rangle + \langle g_y \bar{g} \rangle \langle g \bar{g}_y \rangle \\ \tau_6 &= i \left(\langle g_x \bar{g} \rangle \langle g \bar{g}_y \rangle - \langle g_y \bar{g} \rangle \langle g \bar{g}_x \rangle \right).\end{aligned}$$

- Basis 02:² $\{\tau_1^2, \tau_1 \tau_2, \tau_2^2, \tau_1^2 - \tau_2^2 - 3\tau_7\}$

$$\tau_7 = \langle g_x \bar{g}_x \rangle \langle g_y \bar{g}_y \rangle - \langle g_x \bar{g}_y \rangle \langle g_y \bar{g}_x \rangle.$$

- Basis 21: $\{\tau_3 \tau_1, \tau_3 \tau_2, \tau_4 \tau_1, \tau_4 \tau_2, \tau_5 \tau_1, \tau_5 \tau_2, \tau_6 \tau_1, \tau_6 \tau_2, \tau_8, \tau_9, \tau_{10}, \tau_{11}, \tau_{12}, \tau_{13}\}$

$$\begin{aligned}\tau_8 &= \frac{1}{4}(\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2)\xi_5 + \frac{1}{2}(\xi_1 \xi_2 + \xi_3 \xi_4)\xi_6 \\ \tau_9 &= \frac{1}{8}(\xi_1^2 - \xi_2^2 - \xi_3^2 + \xi_4^2)\xi_5 + \frac{1}{4}(\xi_1 \xi_2 - \xi_3 \xi_4)\xi_6 \\ \tau_{10} &= \frac{1}{4}(\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2)\xi_6 - \frac{1}{2}(\xi_1 \xi_2 + \xi_3 \xi_4)\xi_5 \\ \tau_{11} &= \frac{1}{8}(\xi_1^2 - \xi_2^2 - \xi_3^2 + \xi_4^2)\xi_6 - \frac{1}{4}(\xi_1 \xi_2 - \xi_3 \xi_4)\xi_5 \\ \tau_{12} &= \frac{i}{2}(\xi_1 \xi_3 - \xi_4 \xi_2)\xi_5 + \frac{i}{2}(\xi_4 \xi_1 + \xi_2 \xi_3)\xi_6 \\ \tau_{13} &= \frac{i}{2}(\xi_1 \xi_3 - \xi_4 \xi_2)\xi_6 - \frac{i}{2}(\xi_4 \xi_1 + \xi_2 \xi_3)\xi_5.\end{aligned}$$

- Basis 40:³ $\{\tau_3^2 + \tau_4^2, \tau_3^2 - \tau_4^2, \tau_3 \tau_4, \tau_3 \tau_5, \tau_4 \tau_5, \tau_3 \tau_6, \tau_4 \tau_6, \tau_5^2 + \tau_6^2, \tau_5 \tau_6\}$.

Appendix B. List of the simplest doublets

The simplest doublets for each of the two sets in section 3.8 are obtained by combining the elements of the set into quantities that are eigenfunctions of complex conjugation and mirror transformation, and then pairing them off according to how they transform under the SO(2) coordinate rotations. Here we list the simplest doublets (denoted by Δ_{\square}) for each of the sets. The elements of the doublets are denoted by ξ_{nS} , which are used in some of the expressions in appendix A.

- Set 1: spin-1: $\Delta_1 = [\xi_1, \xi_2]^T$ and $\Delta_2 = [\xi_3, \xi_4]^T$,

$$\begin{aligned}\xi_1 &= \langle g_x \bar{g} \rangle + \langle g \bar{g}_x \rangle + \langle g_y \bar{g} \rangle + \langle g \bar{g}_y \rangle \\ \xi_2 &= \langle g_x \bar{g} \rangle + \langle g \bar{g}_x \rangle - \langle g_y \bar{g} \rangle - \langle g \bar{g}_y \rangle \\ \xi_3 &= \langle g_x \bar{g} \rangle - \langle g \bar{g}_x \rangle + \langle g_y \bar{g} \rangle - \langle g \bar{g}_y \rangle \\ \xi_4 &= \langle g_x \bar{g} \rangle - \langle g \bar{g}_x \rangle - \langle g_y \bar{g} \rangle + \langle g \bar{g}_y \rangle.\end{aligned}$$

- Set 2: spin-2: $\Delta_3 = [\xi_5, \xi_6]^T$,

$$\begin{aligned}\xi_5 &= \langle g_x \bar{g}_y \rangle + \langle g_y \bar{g}_x \rangle \\ \xi_6 &= \langle g_x \bar{g}_x \rangle - \langle g_y \bar{g}_y \rangle.\end{aligned}$$

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² Note that τ_7 is not an orthogonal singlet. Its definition is made for the sake of simplicity, because of the way it appears in the expression of the determinant (32).

³ All the basis elements of basis 40 are reducible.

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