On Equivalent Radius of Curvature for PWL Geometrical Modeling of a Loop Antenna

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***Abstract*—A circular loop antenna is often numerically modeled using a regular polygon. This approach is simple and robust, yet it alters the circumference of the loop and may thus shifts the resonance frequency in the numerical model. This paper introduces a simple analytical formula for predicting and relating the accuracy of resonance frequency determination to the number of segments used. The result of testing on commercial software WIPL-D showed excellent match between the prediction and numerically derived results. It is expected that the approach can be extended onto other structures with curvatures.**

***Index Terms*—Approximation error, electromagnetic modeling, helical antennas, loop antennas, mesh generation, numerical analysis, numerical models, piecewise linear approximation, resonance, thin wire approximation, wire modeling.**

I. Introduction

MANY antenna types are based on or include curved elements or structures, and much work has been dedicated to related modeling [1]-[7]. Examples of such antennas include loop and spiral antennas. Modeling of these two types of antennas can be done theoretically [7], as well as numerically [1]-[5], frequently using a piecewise linear (PWL) approximation of the actual geometry [8]. Solving more complex geometries normally relies on a purely numerical solution, also based on PWL approximation of the geometry, an approach used in the majority of commercial software for computational electromagnetic modeling. The density of the mesh and positioning of its nodes define the accuracy of representing the geometry. Usually, a higher density of mesh produces a higher accuracy and better conditioning of the numerical solution [8], [9]. The required density of the mesh is typically determined based on experience, or automatically, in an iterative manner, observing convergence of the solution. The position of nodes in a complex model is determined in the same way. Such an approach is universal yet suboptimal, due to wasted iterations. It is desirable to be able to minimize the number of convergence iterations without setting the accuracy of the model to an unnecessarily high level.

This paper uses a thin wire model of a circular loop antenna to establish a relationship connecting the density of mesh to the accuracy of the solution, more specifically to the resonance frequency of the loop. For simplicity, it is assumed that the nodes are positioned equidistantly, at the corners of a regular polygon drawn into the circle, as illustrated in Fig. 1.

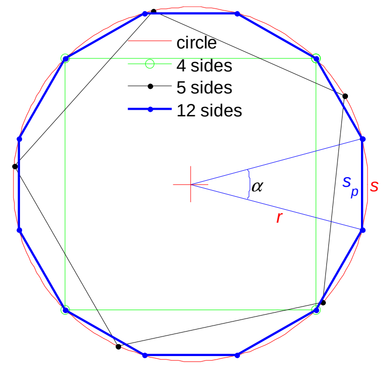

Fig. 1. Geometrical approximation of a circular loop with a regular polygon. Several polygons with four, five and twelve sides are shown as drawn into the circle. The notations in the figure are: r is the circle radius, α is the opening angle of a sector with arc length s, and area A, and s_p is the length of the base of an equilateral triangle fitted into the sector.

The paper is organized as follows. Section II describes the model, assumptions and methodology. The approximation roughness and computation error are analyzed in Section III. The results are then validated in Section IV.

II. Method and Model

A circular loop antenna may be modeled using straight wire segments, i.e. under a piecewise linear (PWL) geometrical approximation, used to generate a regular polygon. Herein, the polygon is assumed to be drawn *into* the circle.

The derivations made below consider a loop antenna made of thin perfectly conducting wire situated in free space. Taking into account the high frequency of an application, it is also assumed that the circumference of the wire in the loop plays a more significant role in the error than the area of this loop.

Fig. 1 shows a set of sample regular n-corner polygons drawn into a circle representing a loop antenna. Assuming that the polygon's centre is at the origin, the coordinates of its vertices may, for example, be calculated as

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$$\begin{cases} \varphi_k = 2\pi k/n, \\ x_k = r\cos(\varphi_k), \\ y_k = r\sin(\varphi_k), \end{cases} \text{ where } k=0,1,...n-1, \quad (1)$$

where r is the radius of the circle/loop, φ_k is angular coordinate of the k-th vertex, and x_k and y_k are the respective rectangular coordinates.

The first resonant frequency f_1 for a circular loop antenna may be derived, approximately [7], using

$$f_1 = \frac{c}{C_c} = \frac{c}{2\pi r}, \quad (2)$$

where c is the speed of light in medium, and C_c is the circumference of the circular loop. In numerical models using a polygon instead of a circle, a straightforward use of the approximation (2) with the polygon drawn into the circle, will lead to a shift in the resonant frequency computed from the numerical model. This occurs due to a decrease in the perimeter for the approximating polygon compared to the circumference of the original circle. Table 1 based on [10], shows a comparison of the main geometrical parameters of a circle and an equivalent regular n-corner polygon.

Table 1. Expressions for the parameters of a circle with radius r, compared against the expressions for respective parameters of a regular n-corner polygon with outer radius r. Fig 1 illustrates some of the parameters.

Geometry, Parameter	**Circle**	**Regular polygon with n corners**
Length of an arc (s) / polygon side with opening angle α (s_p)	$S = \alpha r$	$s_p = 2r\sin(\alpha/2)$
Circumference (C_c) / perimeter (C_p)	$C_c = 2\pi r$	$C_p = n\, s_p = 2\, n\, r\sin(\alpha/2)$

It is possible to minimize this shift in the resonance frequency, by a slight increase in the outer diameter or radius of the approximating polygon in the numerical model. In order to find how much of increase is necessary, the circle's circumference can be set equal to the perimeter of the regular polygon (the subscript $_p$ is used to denote quantities related to the polygon):

$$(C_c = 2\pi r) = \left(C_p = n2r_p\sin(\tfrac{\alpha}{2})\right). \quad (2)$$

This gives the desired *equivalent radius* of the polygon r_p (the one giving a more accurate estimate for the resonance frequency), as a function of the radius of the loop antenna r:

$$r_p = r\frac{\pi}{n\sin(\frac{\alpha}{2})} = r\cdot F_p. \quad (3)$$

In the last expression, the factor F_p was introduced to signify the relative difference in the radius of a circle and the outer radius of a regular polygon. The expression for the factor F_p can be written as

$$F_p = \frac{\pi}{n\sin(\frac{2\pi/n}{2})} = \frac{\pi}{n\sin(\frac{\pi}{n})} = \frac{\alpha/2}{n\sin(\alpha/2)}, \quad (4)$$

where α is the opening angle for a side of the polygon.

Numerically, the value of F_p starts from approximately 1.21 for n=3 and, as the quality of the geometrical approximation increases with n, it asymptotically approaches unity, as n^{-2}.

At lower frequencies, where the loop area dependent magnetic interactions may dominate, similar derivations can be made, starting with equating the areas instead of circumference and perimeter. It can then be found that the radius correction factor for area for n=3 is about 1.56, and so the error in applying the expressions derived in the following section at the below-resonance frequencies could be as much as 1.56/1.21=1.3 times larger.

III. Error due to PWL approximation of geometry

It is also possible to evaluate the relative error in the resonance frequency of a loop antenna. This error may be defined as:

$$\varepsilon = \frac{f_p - f_c}{f_c}, \quad (5)$$

where f_c is the resonance frequency of a circular loop given by (2), and f_p is the resonance frequency of a respective polygonal loop defined similarly, via the perimeter of the polygon C_p: $f_p = c/C_p$. Hereinafter, the resonance frequency of the loop is assumed to be defined by its perimeter only, i.e. the phase velocity of a wave travelling around the loop periphery is assumed to be independent of the geometrical properties of the periphery, including the radius of curvature. Using (5) and Table 1, the expression for the relative error (6) may then be expressed through the factor F_p:

$$\varepsilon = \frac{c/C_p - c/C_c}{c/C_c} = F_p - 1 = \frac{\pi/n}{\sin(\pi/n)} - 1, \text{ or}$$

$$\frac{\sin(\pi/n)}{\pi/n} = \frac{1}{1+\varepsilon}. \quad (6)$$

In order to obtain the minimum number of required segments, n, from the prescribed accuracy of solution ε, one would need to solve this transcendent equation. Solving a transcendent equation might be inconvenient. Instead, it is possible to obtain a simple approximate solution. The expression (6) can be expanded asymptotically for large n by using Taylor's series expansion with parameter $1/n$. Keeping only the first two terms in the expansion of the sine function results in the asymptotic estimate given by

$$\varepsilon \cong \frac{\pi^2}{(3!)n^2 - \pi^2}, \quad n >> 1. \quad (7)$$

Thus, in order to achieve the relative frequency error below ε, the number of segments in a polygonal loop must be equal to or greater than the nearest higher integer of

$$N_\varepsilon = \max\left(3, \frac{\pi}{\sqrt{6}}\sqrt{\frac{1}{\varepsilon}+1}\right), \text{ valid for } \varepsilon << 1 \quad (8)$$

The exact (6) and asymptotic (8) forms may be used to estimate the required quality of geometrical approximation based on the desired accuracy in the resonance frequency. To keep the same level of accuracy for frequencies different to the first resonance frequency, the π in right hand side in (9) will need to be additionally factored with the ratio of the required frequency to the first resonance frequency, increasing the number of wire segments used in proportion to the frequency.

IV. Numerical Validation

The expressions (7) and (9) have been applied to a polygonal loop antenna modeled using various number of wire segments. To compare these results against a numerical model, the resonance frequency error was also obtained from the input reactance data computed using WIPL-D [8] applied to a circular loop antenna modeled with a regular polygon having the number of sides varying from 3 to 39, and made of thin wire with radius $1.64 \cdot 10^{-3}\lambda$. The number 39 was chosen to preserve the thin wire approximation [8]. The accuracy level for impedance matrix elements calculation was set to maximum (level 10) to minimize the unrelated effects. The degree of current-approximating-polynomial was set to automatic. Sample WIPL-D model geometries are shown in Fig. 2.

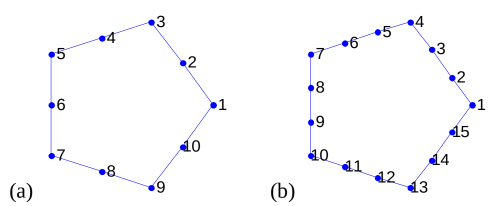

Fig. 2. Samples WIPL-D models: two regular polygons with 5 sides, modeled using (a) 1 sub-division of each side into two segments and (b) 2 sub-divisions (into 3 segments).

The resonance frequency, defined as the frequency corresponding to the point of zero input reactance, was then extracted from each WIPL-D model and compared against the most accurate WIPL-D model computed with 39 segments.

The results are shown in Fig. 3. The theoretical curves run very close to and in parallel to the numerical estimations, indicating an excellent match of the theoretical solution derived. The figure also demonstrates that (i) the expression (9) is very close to its more accurate counterpart (7), even for the lowest practical n=3, and that (ii) multiplying the π in the right hand side of (9) by an empirically determined factor of about 1.1 can match the theoretical and numerical results even better. In terms of the predictions, it may be also important to factor in the influence of the current-approximating function, which, as may be seen from Fig. 3, may require an additional factor with value of up to 1.4.

There are also some artifacts visible in Fig. 3: (a) the crossing of the theoretical and numerical solutions around n=20 was found to be due to the use of the reference resonance frequency determined from a solution of limited accuracy (n=39); (b) the different behavior of the "0 sub-divisions" curve is due to the change of the current-approximating polynomial from the 2nd to 1st degree, triggered by the WIPL-D's automatic routines, and effected from n=7.

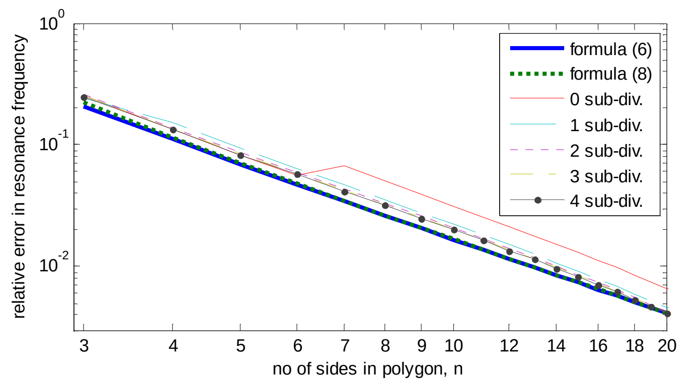

Fig. 3. Error in determining the first resonance frequency of a circular loop antenna modeled with a regular polygon of n sides/corners, using formulas (7) and (9), and WIPL-D. The latter computed results for different number of sub-divisions per the side of a polygon to illustrate numerical convergence, e.g. "0 sub-div." means that only 1 wire was used to connect the corners.

V. Conclusion

A geometrical approximation of a circular loop antenna by a regular polygon, suitable in numerical modeling, has been considered. The error in determining the first resonance frequency of the loop antenna made of thin wire was used to derive the relationship between the accuracy of the resonance frequency and the number of polygon corners required to obtain such accuracy.

The estimations derived are expected to be also valid for other wire antennas with curvatures, such as helical antennas, spiral antennas (per radiating region's radius) and surface based elements including higher modes on plates, as long as the other dimensions of the curvature are negligible.

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