

High-Order Noise Filtering in Nontrivial Quantum Logic Gates

Todd Green,¹ Hermann Uys,² and Michael J. Biercuk^{1,*}

¹Centre for Engineered Quantum Systems, School of Physics, The University of Sydney, New South Wales 2006 Australia

²National Laser Centre, Council for Scientific and Industrial Research, Pretoria, South Africa

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Treating the effects of a time-dependent classical dephasing environment during quantum logic operations poses a theoretical challenge, as the application of noncommuting control operations gives rise to both dephasing and depolarization errors that must be accounted for in order to understand total average error rates. We develop a treatment based on effective Hamiltonian theory that allows us to efficiently model the effect of classical noise on nontrivial single-bit quantum logic operations composed of arbitrary control sequences. We present a general method to calculate the ensemble-averaged entanglement fidelity to arbitrary order in terms of noise filter functions, and provide explicit expressions to fourth order in the noise strength. In the weak noise limit we derive explicit filter functions for a broad class of piecewise-constant control sequences, and use them to study the performance of dynamically corrected gates, yielding good agreement with brute-force numerics.

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Dynamical error suppression strategies have been demonstrated as a means by which errors due to decoherence may be suppressed during qubit memory operations [1–8]. In such cases, a filter-design framework [9–13] has successfully shown how to precisely estimate average error rates in the presence of classical, time-dependent noise. Expanding this analysis beyond the identity operator has proved challenging due to the need for efficient techniques to treat a random, time-varying noise term that does not commute with the applied control field. Understanding the influence of such time-dependent processes during control operations is vital, however, as environmental decoherence sets the lower-bound on achievable gate error rates in quantum informatic settings, and the residual uncertainty in the operation of atomic clocks; and can limit the performance of quantum-enabled sensors.

In this Letter we address the challenge of characterizing and mitigating decoherence due to classical noise during nontrivial single-qubit operations. We calculate the ensemble-averaged entanglement fidelity for an arbitrary control sequence to fourth order in the noise strength, incorporating terms to the third order of the Magnus expansion in an effective Hamiltonian treatment. For concreteness we explicitly calculate the filter functions to lowest nontrivial order for sequences composed of π rotations about Cartesian axes with arbitrary rotation rates—a class including dynamically corrected gates (DCGs). Our results permit detailed calculation of the generic (amplitude and phase) errors that result from applying a quantum control operation in the presence of pure-dephasing noise, and validate perturbative predictions of an increased order of error suppression [14,15]. This straightforward analytical approach is compared against brute-force numerical calculations of the evolution of the Bloch vector, and shows excellent agreement.

We consider the canonical dephasing environment in which the system evolves freely under a Hamiltonian of the form $H = \frac{1}{2}[\Omega + \eta(t)]\sigma_z$, where Ω is the unperturbed qubit splitting, η is a time-dependent classical random variable, and σ_z is a Pauli operator. In the case of free evolution the presence of a nonzero $\eta(t)$ produces dephasing in an ensemble average. However, during driven operations where one applies a control field proportional to σ_x or σ_y , the presence of a pure-dephasing noise environment yields both polarization damping and dephasing effects. Both must be considered in a full treatment of gate errors. This accounts for the most significant *correctable* forms of decoherence in experiments; most remaining polarization-damping errors are due to stochastic processes that cannot be corrected through dynamical error suppression.

We begin with an outline of our method. The total Hamiltonian (in the rotating frame at Ω) is $H(t) = H_0(t) + H_c(t)$, over $t \in [0, \tau]$. The operator $H_0(t) = \eta(t)\sigma_z/2$ represents a time-varying dephasing environment, while $H_c(t)$ describes an interaction between the system and an external control device that, in principle, may be used to implement arbitrary rotations of the Bloch vector. In general, evaluating the total propagator $U(t) = T \exp(-i \int_0^t H(t') dt')$ explicitly for $H_0(t) \neq 0$ is difficult due to noncommuting terms in $H(t)$. We therefore proceed by factoring out that part of the qubit evolution that is due solely to the control and expressing the residual ‘error propagator’ $\tilde{U}(\tau)$ in terms of a *time-independent* effective Hamiltonian that can then be evaluated, following a general procedure laid out in Ref. [2].

Defining the control propagator $U_c(t) = T \exp(-i \int_0^t H_c(t') dt')$, it can be shown that $\tilde{U}(t) = U_c^\dagger(t)U(t)$ satisfies the equation of motion $i \frac{d\tilde{U}(t)}{dt} = \tilde{H}_0(t)\tilde{U}(t)$, where

$\tilde{H}_0(t) \equiv U_c^\dagger(t)H_0(t)U_c(t)$ [16,17]. If $H_c(t)$ enacts a target unitary operation Q in the absence of decoherence, then we may write the total noise-affected operation as $U(\tau) = Q\tilde{U}(\tau)$. The error propagator can then be expressed in terms of a time-independent effective Hamiltonian H_{eff} , defined by $\tilde{U}(\tau) \equiv \exp(-iH_{\text{eff}}\tau)$.

For a dephasing Hamiltonian we have $\tilde{H}_0(t) = \frac{\eta(t)}{2}U_c^\dagger(t)\sigma_z U_c(t)$; the term $U_c^\dagger(t)\sigma_z U_c(t)$ is a rotation of the σ_z operator due to the control. We may therefore define a time-dependent ‘control vector’ in a Cartesian basis ($l = x, y, z$), $\vec{s}_1(t) = \sum_l s_{1,l}(t)\hat{l}$, where $|\vec{s}_1(t)| = 1$, $\forall t \in [0, \tau]$, such that $\tilde{H}_0(t) = \frac{\eta(t)}{2}\vec{s}_1(t) \cdot \vec{\sigma}$. Since $\tilde{H}_0(t)$ belongs to the Lie algebra $su(2)$, $\forall t \in [0, \tau]$, the effective Hamiltonian H_{eff} derived from it is also in $su(2)$. We may now write $U(\tau) = Q \exp[-i\vec{a}(\tau) \cdot \vec{\sigma}]$, where the effect of the noise on the ideal operation Q is encoded in the ‘error vector’ $\vec{a}(\tau)$. Here $\vec{a}(\tau) = \sum_l a_l(\tau)\hat{l}$ represents error contributions to Q due to terms in H_{eff} proportional to the Cartesian components of the Pauli operator. In general, the action of the control converts pure-dephasing noise to arbitrary rotations of the qubit’s Bloch vector, producing both dephasing and polarization damping errors.

In most cases, an explicit expression for $\exp[-i\vec{a}(\tau) \cdot \vec{\sigma}]$ can only be found using approximation methods. For our purposes, it is desirable that the simple exponential form of the operator be retained to any level of approximation. This can be achieved through the use of the Magnus expansion [18,19]. Using this method, the exponent is expanded in an infinite series of time integrals over nested commutators of $\tilde{H}_0(t)$ at different times. In conjunction with the identity $[\vec{u} \cdot \vec{\sigma}, \vec{v} \cdot \vec{\sigma}] = 2i(\vec{u} \times \vec{v}) \cdot \vec{\sigma}$, $\vec{u}, \vec{v} \in \mathbb{R}^3$, we find that we can write $\vec{a}(\tau) \cdot \vec{\sigma} = \sum_{i=1}^{\infty} \vec{a}_i \cdot \vec{\sigma}$ to all orders i . The first three terms in series expansion of the error vector are $\vec{a}_1(\tau) = \frac{1}{2} \int_0^\tau dt \eta(t)\vec{s}_1(t)$, $\vec{a}_2(\tau) = \frac{1}{4} \int_0^\tau dt_2 \int_0^{t_2} dt_1 \eta(t_1)\eta(t_2)\vec{s}_2(t_1, t_2)$, and $\vec{a}_3(\tau) = \frac{1}{12} \int_0^\tau dt_3 \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \eta(t_1)\eta(t_2)\eta(t_3)\vec{s}_3(t_1, t_2, t_3)$. Here, high-order commutators in the Magnus expansion have been reduced to vector cross products of the control vector at different times, $\vec{s}_2(t_1, t_2) \equiv \vec{s}_1(t_2) \times \vec{s}_1(t_1)$, and $\vec{s}_3(t_1, t_2, t_3) \equiv \vec{s}_1(t_3) \times [\vec{s}_1(t_2) \times \vec{s}_1(t_1)] + [\vec{s}_1(t_3) \times \vec{s}_1(t_2)] \times \vec{s}_1(t_1)$.

We evaluate the net effect of the gate operation Q in the presence of noise via the ensemble average entanglement fidelity [20], $\langle \mathcal{F}(\tau) \rangle = \langle |\text{Tr}(QU(\tau)/2)|^2 \rangle$ which, when written in terms of the error vector, becomes $\langle \mathcal{F}(\tau) \rangle = \frac{1}{2}[\langle \cos[2|\vec{a}|] \rangle + 1]$. To evaluate the fidelity we write $|\vec{a}| = (\sum_l a_l^2)^{1/2}$, express the cosine term as a Taylor series, and substitute the Magnus expansion $a_l = \sum_{i=1}^{\infty} a_{i,l}$ for each component of the error vector. The result is an infinite series of multidimensional integrals over products of multiple-time noise correlation functions and components of the control vector $\vec{s}_1(t)$. For example, the lowest-order effects of the noise are captured by $\langle a_{1,l}^2 \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s_{1,l}(t_1)s_{1,l}(t_2)\langle \eta(t_1)\eta(t_2) \rangle dt_1 dt_2$, for $l = x, y, z$.

Assuming the noise is Gaussian, only correlation functions $\langle \eta(t_1) \dots \eta(t_n) \rangle$ for which n is even contribute. Further, applying the Gaussian moment theorem, each of these can be written in terms of simple *two-point* correlation functions. Using the root mean square deviation of the noise, $\Delta\eta \equiv \sqrt{\langle \eta(t)^2 \rangle}$, as a measure of the noise strength, we can define a parameter $\xi \equiv \Delta\eta\tau/2$ which provides an upper bound for the magnitude of each term in the series expansion of the cosine function [21,22]. If we restrict our analysis to weak-noise or efficient-control conditions under which $\xi < 1$, then higher-order terms provide diminishing contributions to the total error and we may truncate the series. To fourth order in ξ we find that

$$\begin{aligned} \langle \cos[2|\vec{a}|] \rangle &= 1 - 2\xi^2 \overline{\langle a_{1,l}^2 \rangle} \\ &\quad - 2\xi^4 \left\{ \frac{1}{4} (3\overline{\langle a_{2,l}^2 \rangle} + 2\overline{\langle a_{1,l}a_{3,l} \rangle}) - \overline{\langle a_{1,l}^2 a_{1,l'}^2 \rangle} \right\}, \end{aligned} \quad (1)$$

where summations are implicitly performed over l and l' . An overline indicates that the maximum value of the term has been factored out (e.g., $\max(|\langle a_{1,l}^2 a_{1,l'}^2 \rangle|) = 3\xi^4$, $\therefore |\overline{\langle a_{1,l}^2 a_{1,l'}^2 \rangle}| \equiv |\langle a_{1,l}^2 a_{1,l'}^2 \rangle|/(3\xi^4) \leq 1$). Equation (1) includes terms to *third* order in the Magnus expansion as they contribute to the same order in ξ as the second-order term (see Supplementary Material [30]).

The various contributions to Eq. (1) may be calculated explicitly by Fourier transforming the noise and control. For instance, we may write

$$\langle a_{1,l}^2 \rangle = \frac{1}{4\pi} \int_0^\infty |y_{1,l}(\omega)|^2 \frac{S(\omega)}{\omega^2} d\omega. \quad (2)$$

with $y_{1,l}(\omega) = -i\omega \int_{-\infty}^{\infty} s_{1,l}(t)e^{i\omega t} dt$, capturing terms proportional to σ_l . We sum over l to write $F_1(\omega) = \sum_l F_{1,l} = \sum_l |y_{1,l}(\omega)|^2$, corresponding to the terms in the first line of Eq. (1). Similarly, we account for the proliferation of higher-order terms arising from the vector cross product by defining $F_{p,2}(\omega, \omega', \tau)$, where p is an index over terms proportional to ξ^4 in Eq. (1). These terms contain four-point correlation functions in time, $\langle \eta(t_1)\eta(t_2)\eta(t_3)\eta(t_4) \rangle$, which may be explicitly evaluated using the Gaussian moment theorem.

We may then compactly write the entanglement fidelity

$$\begin{aligned} \langle \mathcal{F} \rangle &= 1 - \frac{1}{4\pi} \int_0^\infty \frac{d\omega}{\omega^2} S(\omega) F_1(\omega, \tau) - \frac{1}{(4\pi)^2} \sum_p \int_0^\infty \frac{d\omega}{\omega^2} S(\omega) \\ &\quad \times \int_0^\infty \frac{d\omega'}{\omega'^2} S(\omega') F_{p,2}(\omega, \omega', \tau). \end{aligned} \quad (3)$$

Terms to all orders in ξ may be evaluated using the same procedure.

With these expressions we have reduced the effect of a time-dependent dephasing environment and a

time-dependent control Hamiltonian to integrals incorporating only stationary statistical properties of the system: the noise power spectral density and the spectral functions to arbitrary order, F_i . These terms contain all relevant information about the applied control, and the explicit inclusion of terms proportional to all σ_l captures both dephasing and polarization-damping errors produced during control operations. We refer to the terms F_i as the *filter functions* for the total control operation, in analogy with previous work on dynamical decoupling [10–13]. The leading nontrivial term is closely related to the idea of spectral overlap functions between control and noise that has been studied previously [23–25], but a generalized derivation to high order has not appeared to the best of our knowledge.

We now consider a specific case for concreteness that is germane to many coherent control experiments, including the implementation of dynamically corrected gates. We define $H_c(t)$ as piecewise constant over a total of $k \in \mathbb{N}$ consecutive time bins, such that during the j th time bin the control is intended to execute σ_{l_j} , restricted here to either the identity I , or a rotation of the qubit Bloch vector through $\pm\pi$ about one of the three Cartesian directions. In this notation, $l_j = I, x, y, z$.

The control propagator may be written explicitly such that during the j th driven operation we have $U_c^{(j)} = \exp[-i\Omega_R^{(j)}(t - t_{j-1})\sigma_{l_j}/2]\sigma_{\forall j-1}$, where $\Omega_R^{(j)}$ gives the driven rotation rate about axis \hat{l} in time bin j . The operator $\sigma_{\forall j-1} = \sigma_{l_{j-1}}\sigma_{l_{j-2}}\dots\sigma_{l_1}$ describes the cumulative effect of all completed rotations in the preceding time segments.

We restrict our presentation to terms in the entanglement fidelity of order ξ^2 , appropriate for the case of weak dephasing noise. Higher-order contributions are straightforward to calculate explicitly, but involve many dozens of terms. In this case we only require first-order components of the error vector and may approximate $\langle \mathcal{F}(\tau) \rangle \approx \frac{1}{2}[e^{-2\sum \langle a_{i,l}^2 \rangle} + 1]$. Using the first-order filter functions we have $\langle \mathcal{F}(\tau) \rangle \approx \frac{1}{2}[\exp(-\chi(\tau)) + 1]$, where $\chi(\tau) = \frac{1}{2\pi} \int_0^\infty F_1(\omega) \frac{S(\omega)}{\omega^2} d\omega$.

For piecewise-constant control as described above, the terms contributing to $F_{1,l}(\omega)$ may be written

$$\begin{aligned}
 y_{1,x}(\omega) &= \sum_{j=1}^k (-1)^{N_{y,z}^{[j]}+1} \frac{i\omega\Omega_R^{(j)}}{\omega^2 - (\Omega_R^{(j)})^2} (e^{i\omega t_j} + e^{i\omega t_{j-1}}) \delta_{l_j y}, \\
 y_{1,y}(\omega) &= \sum_{j=1}^k (-1)^{N_{x,z}^{[j]}} \frac{i\omega\Omega_R^{(j)}}{\omega^2 - (\Omega_R^{(j)})^2} (e^{i\omega t_j} + e^{i\omega t_{j-1}}) \delta_{l_j y}, \\
 y_{1,z}(\omega) &= \sum_{j=1}^k (-1)^{N_{x,y}^{[j]}} \left[\frac{\omega^2}{\omega^2 - (\Omega_R^{(j)})^2} (e^{i\omega t_j} + e^{i\omega t_{j-1}}) \right. \\
 &\quad \left. \times (\delta_{l_j x} + \delta_{l_j y}) + (e^{i\omega t_{j-1}} - e^{i\omega t_j})(\delta_{l_j z} + \delta_{l_j l}) \right] \quad (4)
 \end{aligned}$$

where $\delta_{\alpha\beta} = 1$ for $\alpha = \beta$ and is zero otherwise. Here $N_{\alpha,\beta}^{[j]}$ represents the number of times σ_α or σ_β appear in the sequence up to the j th interval. We note that the prefactors in these equations are similar to those derived from the steady-state master-equation treatment of a driven two-level system in the presence of dissipation [26].

Using this approach we are able to analyze the effects of classical noise on a complete set of both primitive (standard) and dynamically protected single-qubit gates of interest for quantum logic. As an example, in Fig. 1 we show the total filter functions derived from this treatment for X operations in primitive and DCG formulations [14,27]. Consistent with previous perturbative treatments of quantum-mechanical baths, we find that the order of error suppression, given by the low-frequency rolloff of $F_1(\omega)$, is increased in the DCG relative to the primitive gate [13,28]. The extension of the duration of X_{DCG} relative to X is manifested as a decrease in ω_{F1} , the frequency above which the filter function takes the value unity and noise is passed largely unimpeded. By examining the phase and amplitude components of the filter function independently ($F_{1,z}$ and $F_{1,x}$), we see that for a DCG we improve the order of error suppression primarily in the term that commutes with the control operation (*e.g.*, amplitude for an X gate).

The form of the filter functions for other operations and their DCG constructions (*e.g.*, Z_θ , H) show similar behavior. Dynamical decoupling (dynamically protected I) may also be treated using Eq. (4), and will be addressed in detail in a separate manuscript.

Using the lowest-order approximation for the entanglement fidelity and the specific forms of the filter functions for X_{DCG} we calculate the error probability for different noise environments and plot these in Fig. 2. As expected, in an environment given by $S(\omega) = \alpha/\omega^2$, with α a scaling factor (*cf.* Ref. [29]), we find significant benefits from using the DCG construction. By contrast, in a white noise

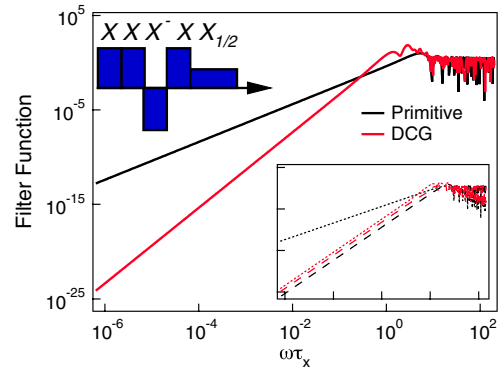


FIG. 1 (color online). Filter functions for primitive X (shallow line) and X_{DCG} (steep line), based on Eq. (4). Upper inset, schematic construction of X_{DCG} [14]. X^- is an X rotation with π phase shift. Lower inset, amplitude, $F_{1,x}$ (dotted), and phase $F_{1,z}$ (dashed) filter functions for the same gates, denoted by color. Tick marks are the same as in the main panel.

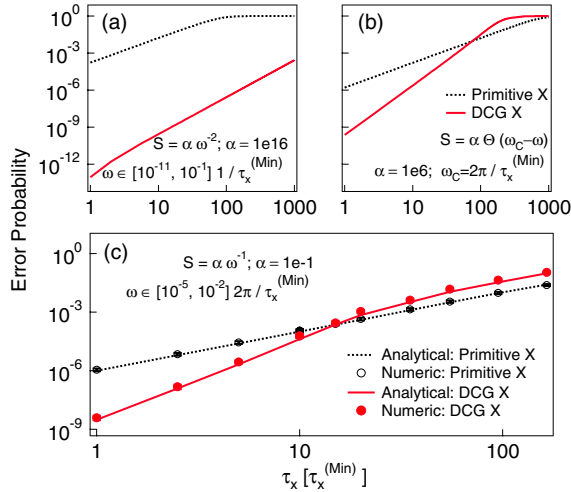


FIG. 2 (color online). Calculated error rates for primitive X and X_{DCG} gates in the presence of noise. (a) Calculated error rates in the presence of noise similar to that observed in Ref. [29], appropriate for decoherence due to nuclear spin diffusion in solid-state singlet-triplet qubits. Data are plotted in dimensionless units of the minimum τ_x value. (b) Similar to (a), but using a white noise power spectral density up to a sharp high-frequency cutoff. (c) Comparison of calculated error rates based on analytical filter functions derived herein and brute-force numerics. Noise strength in numerics is set using $\Delta = 0.008[\tau_x^{(\text{Min})}]^{-1}$, guaranteeing convergence of the Magnus expansion for this range of τ_x . Each data point averages over 50 randomly generated noise trajectories whose statistical properties reproduce $S(\omega)$. Error bars are derived from the root-mean-square deviation of the individual trajectory results, but are small compared to the marker size.

environment with a sharp high-frequency cutoff [$S(\omega) = \alpha \Theta(\omega_c - \omega)$] the significant high-frequency spectral components of the noise and the extended duration of the DCG construction can yield a net performance degradation in the event of long τ_x and large ω_c . The relationship between these two quantities thus provides a dominant practical limit on the applicability of DCG construction in realistic settings; so long as $\omega_c \lesssim \tau_x^{-1}$ the DCG provides net performance enhancement.

Validation for the assumptions and approximations made in this treatment comes from performing detailed brute-force numerical simulation of the evolution of the Bloch vector in the presence of a noisy environment, and averaging over multiple trajectories, $\eta(t)$ [Fig. 2(c)]. The trajectories are chosen to exhibit the statistical properties of a desired power spectral density. Calculations involving only the terms in Eq. (4) accurately reproduce numerically calculated error probabilities to within $\sim 20\%$ for the X gate. In complex DCG sequences efficient decoupling reduces leading-order terms until residual contributions become comparable to higher-order terms. We observe that the lowest-order filter functions underestimate error and deviation from numerics grows

with more complex sequences or stronger noise, but remains a factor of order unity for the cases we have studied.

In summary we have developed a theoretical treatment permitting the calculation of error rates due to time-varying classical noise during arbitrary control operations, to arbitrary order. We have explicitly produced a general high-order approximation to the qubit's entanglement fidelity using the first three orders of the Magnus expansion. Building on these results we have presented an example case of piecewise-constant control, and given simple, leading-order expressions for the filter function. This allows any experimentalist to evaluate expected error rates to the correct order of magnitude *during nontrivial control operations*, accounting for the fact that pure dephasing noise can result in both dephasing and polarization damping errors during control operations. Included in this class of control sequences are dynamically corrected gates, and we have validated previous perturbative calculations for DCG performance through an intuitive noise-filtering approach.

We believe that this experimentally accessible and rigorously tested method for understanding the influence of classical noise in single-qubit logic operations will prove valuable to the community and will open new pathways for the development of error-robust quantum control strategies. We emphasize that our approach is technology independent and these methods apply to any quantum system.

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*michael.biercuk@sydney.edu.au

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