Lateral Phase Drift of the Topological Charge Density in Stochastic Optical Fields

Filippus S. Roux

CSIR National Laser Centre, PO Box 395, Pretoria 0001, South Africa

Abstract

The statistical distributions of optical vortices or topological charge in stochastic optical fields can be inhomogeneous in both transverse directions. Such two-dimensional inhomogeneous vortex or topological charge distributions evolve in a complex way during free-space propagation. While the evolution of one-dimensional topological charge densities can be described by a linear diffusion process, the evolution of two-dimensional topological charge densities exhibit some additional nonlinear dynamics. Here we propose a phase drift mechanism as a partial explanation for this additional nonlinear dynamics. Numerical results are presented in support of this proposal.

Keywords: optical vortex, singular optics, stochastic optical field, topological charge density

1. Introduction

When a coherent optical beam is scattered from a rough surface the scattered light forms a speckle field. The phase function of such a speckle field contains several phase singularities [1, 2] that propagate along with the field as optical vortices [3, 4]. The handedness of the phase variation around the singularities divide the vortices that are found in such speckle fields into two types according to their topological charge ±1. These two types of vortices are therefore generally referred to as positive and negative vortices. During propagation, pairs of vortices with opposite charges (vortex dipoles) often meet and annihilate each other. On the other hand, vortex dipoles are also created during propagation. The net effect is to keep the vortex density more or less constant at $1/2A_{coh}$, where A_{coh} is the coherence area of the speckle field [1, 5].

The constant vortex density implies that the annihilation rate equals the creation rate. This is reminiscent of a dynamic system in equilibrium, which begs the question, what happens in the non-equilibrium equivalent of a speckle field? To broaden the concept of optical speckle fields to include non-equilibrium scenarios, we'll refer to any optical field that contain some element of randomness as a stochastic optical field. A speckle field, which is also called a random optical field, is then a special stochastic optical field that is in equilibrium.

The vortices in fully developed speckle fields have received much attention [1, 2, 6, 7, 5, 8, 9]. These studies include investigations of the distribution of the optical vortex parameters [7, 8] and also the topology of the vortex trajectories [10, 11].

Here we are more interested in stochastic optical fields that are not in equilibrium. There are various ways to produce non-equilibrium stochastic optical fields. One can start with a speckle field and remove the continuous phase. The resulting field is still homogeneous, but by observing the vortex density, for instance, one can see that the resulting optical field is a non-equilibrium stochastic optical field evolving as a function of the propagation distance toward a new state of equilibrium, which again resembles a speckle field [12, 13]. On the other hand one can generate an artificial stochastic field with inhomogeneous vortex distributions by modulating the phase of a plane wave or a Gaussian beam with a phase-only diffractive optical element or a spatial light modulator that contains a specially designed phase function [14]. Another way to generate such inhomogeneous stochastic fields is through the coherent and incoherent combination of speckle beams [15]. In both these cases one can produce inhomogeneous stochastic fields, which will evolve during propagation, eventually returning to a state of equilibrium where it will resemble a speckle field.

Non-equilibrium stochastic fields exist in many practical situations, such as where a light beam becomes scintillated while propagating though a turbulent atmosphere, or where highly complex diffractive optical elements are used for optical processing. An understanding of the evolution of optical vortices and topological charge distributions can be of great benefit in these applications.

In this paper we study the evolution of stochastic optical fields with two-dimensional inhomogeneous initial topological charge densities. The general aim is to work towards a complete theory for the evolution of such distributions, expressed in terms of differential equations that only contain the vortex distributions under consideration. This work builds on previous work [14] where we investigated the diffusion effect in stochastic optical fields with one-dimensional inhomogeneous initial topological charge densities. The focus of the current paper is a phase drift effect that appears when the initial topological charge density is a two-dimensional function.

The free-space propagation of light is a linear process. However, if the quantity of interest is a nonlinear function of the op-

Email address: fsroux@csir.co.za (Filippus S. Roux)

tical field, as is the case with the vortex distributions considered here, it is not surprising that one would find that such quantities evolve nonlinearly. This is indeed what we find. It is important that we distinguish between the free-space propagation process that deals with optical fields and the, as yet unknown, dynamical equations that govern the evolution of the vortex distributions. In the linear free-space propagation process every plane wave propagates independently. As a result, if the initial complex amplitude that represents the optical field does not contain a particular spatial frequency, then free-space propagation will not generate this spatial frequency in the output. This is a direct result of the linearity of the free-space propagation process. If, on the other hand, we find that new spatial frequencies are generated in the final vortex distribution that were not present in the initial vortex distribution, then this is an unambiguous indication that the dynamics that describe the evolution of the vortex distribution is nonlinear.

The paper is organized as follows. First we provide some background and describe our notation in Sec. 2. The mechanism for phase drift is explained in Sec. 3. Part of the phase drift term contains a convolution process, which is considered in more detail in Sec. 4. In Secs. 5 and 6 we, respectively, provide analytical solutions and numerical simulation results for comparison. The results are discussed in Sec. 7 and we end with some conclusions in Sec. 8.

2. Notation and background

There are two equivalent ways to represent vortex distributions. One way is to represent the positive and negative vortices by their respective number densities, $n_p(x,y,z)$ and $n_n(x,y,z)$. The number density gives the local expectation value for the number of vortices per unit area on any plane perpendicular to the direction of propagation, which is defined by the z-coordinate here. On a given plane the number density varies as a function of the transverse coordinates x and y, and this function can vary from plane to plane as a function of z.

The alternative way to represent the vortex distributions is to define two distinct quantities: the vortex density $V(x,y,z) = n_p(x,y,z) + n_n(x,y,z)$ and the topological charge density $T(x,y,z) = n_p(x,y,z) - n_n(x,y,z)$. The vortex density $V(x,y,z)$ is a positive function just like $n_p(x,y,z)$ and $n_n(x,y,z)$. The topological charge density, on the other hand, can also be negative. The latter is the local expectation value for the net topological charge per unit area on a transverse plane.

Topological charge is locally conserved. As a result one can formulate conservation equations for all the distributions. Considering only vortices with a particular topological charge one finds that they are only conserved up to creation and annihilation events. Therefore one can express their conservation equations as [14]

$$\partial_z n_p(x,y,z) + \nabla\cdot\mathbf{J}_p(x,y,z) = C - \mathcal{A} \tag{1}$$

$$\partial_z n_n(x,y,z) + \nabla\cdot\mathbf{J}_n(x,y,z) = C - \mathcal{A}, \tag{2}$$

where $\mathbf{J}_p(x,y,z)$ and $\mathbf{J}_n(x,y,z)$ are the currents associated with $n_p(x,y,z)$ and $n_n(x,y,z)$, respectively; C and $\mathcal{A}$ respectively represent the local expectation value for the number of creation and annihilation events per unit volume, and $\nabla = \partial_x\hat{x} + \partial_y\hat{y}$.

The equivalent conservation equations for the vortex density and the topological charge density follow directly from Eqs. (1) and (2),

$$\partial_z V(x,y,z) + \nabla\cdot\mathbf{J}_V(x,y,z) = 2C - 2\mathcal{A} \tag{3}$$

$$\partial_z T(x,y,z) + \nabla\cdot\mathbf{J}_T(x,y,z) = 0, \tag{4}$$

where $\mathbf{J}_V(x,y,z) = \mathbf{J}_p(x,y,z) + \mathbf{J}_n(x,y,z)$ and $\mathbf{J}_T(x,y,z) = \mathbf{J}_p(x,y,z) - \mathbf{J}_n(x,y,z)$. The fact that the right-hand side of Eq. (4) is zero, is an expression of the local conservation of topological charge. It also implies that the topological charge density is a simpler quantity to work with and therefore it is the one that we focus on here.

The exact nature of the currents and the creation and annihilation events, and how they depend on the densities are not currently known. As a result, apart from the conceptual picture they provide, the conservation equation in Eqs. (1-4) are not very useful. It is necessary to unravel the dynamics of these densities through detailed investigations of their behavior.

3. Phase drift mechanism

To determine the dependence of the currents on the number densities, one can investigate the motion of individual optical vortices in the beam. We are particularly interested in how the motion of a vortex is influenced during propagation by properties of the beam in the vicinity of the vortex. Fluctuations in the phase and amplitude in the region surrounding a vortex, would in general influence the motion of the vortex. These fluctuations can be caused by other vortices in the region or by the continuous variations in the beam itself.

Inspired by the successes of statistical physics, one can assume that changes in the vortex number densities on the transverse plane can either be produced by the diffusion of the vortices or due to a drift caused by some ‘force.’ One can therefore replace the divergence of the current in Eq. (4) by the sum of a diffusion term and a drift term. The result is a Fokker-Planck equation for the topological charge density.

The diffusion term has, to some extent, been unraveled previously by considering one-dimensional topological charge densities [14]. The resulting diffusion equation, generalized to two-dimensions is given by

$$\partial_z T(x,y,z) - \kappa_0 z\nabla^2 T(x,y,z) = 0, \tag{5}$$

where κ_0 is a dimensionless diffusion parameter and ∇^2 is the transverse Laplacian. Note that this diffusion equation differs from the more familiar equivalent from statistical physics in that the diffusion ‘constant’ is not a constant, but depends linearly on the propagation distance z, as discussed in [14]. The diffusion parameter κ_0 is proportional to λ^2/L_c^2, where λ is the wavelength and L_c is the transverse coherence length ($L_c^2 = A_{coh}$). Although some estimate for the proportionality constant has been made previously [14], we’ll treat this constant as an unknown. The one-dimensional solutions of Eq. (5) are of the form

$$T(x,z) = \cos(2\pi a_0 x)\exp(-2\kappa_0\pi^2 a_0^2 z^2). \tag{6}$$

A drift term is usually associated with an external force that acts on the particles. One can also have interactions of particles on themselves in a collective fashion. There is a way in which optical vortices can produce such a collective self-interaction. In a region where the topological charge density is nonzero the number of optical vortices of one handedness exceeds that of the other. In this case the phase functions of the vortices collectively produce regions of high spatial frequency with particular orientations as shown in Fig. 1. Such a region therefore has a wavefront that is tilted with respect to a plane perpendicular to the general direction of propagation. This tilt causes a sideways drift motion for the vortices in that region. A one-dimensional sinusoidal topological charge density will produce a phase function that contains such regions with high spatial frequencies, as shown in Fig. 1. However, if, for argument sake, the one-dimensional topological charge density is a function of x (as in Fig. 1), then the drift that it produces will be in the direction of y. So the drift will not be able to change the topological charge density. It therefore follows that this drift mechanism does not have any effect in the case of a one-dimensional distribution.

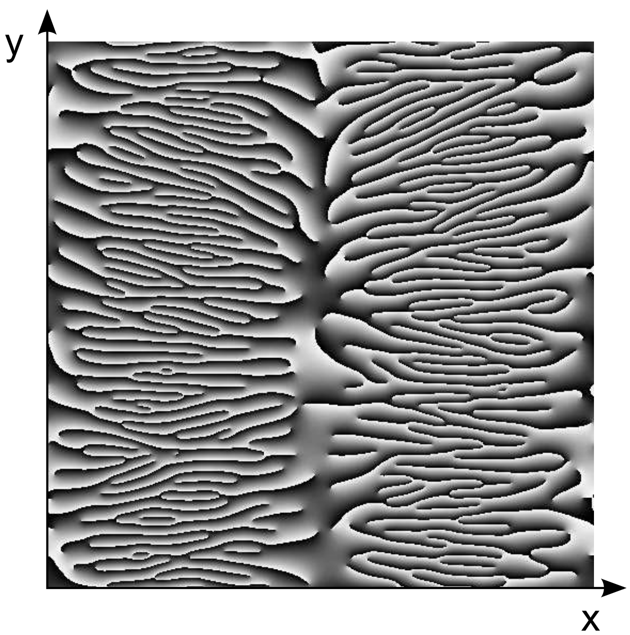

Figure 1: A phase function produced by a one-dimensional sinusoidal topological charge density along the x-axis. Note the high spatial frequencies generated between regions of opposite topological charges.

The influence of phase on the motion of a vortex is determined by the general observation that any wavelet tends to propagate perpendicular to the local wavefront. Consider for instance a beam with a complex amplitude given by

$$f(x,y,z) = A(x,y,z)\exp[i\theta(x,y,z)], \tag{7}$$

propagating in the direction of the propagation vector $\mathbf{k}$, which makes a small non-zero angle with the z-axis, as shown in Fig. 2. Due to this small angle the beam will shift laterally during propagation. If the complex amplitude is given by $f(x,y,z_0)$ at $z = z_0$ then after a small distance of propagation at $z = z_0 + \Delta z$ this function will look more or less the same apart from a lateral shift. Hence, $f(x,y,z_0+\Delta z) \approx f(x-\Delta x, y-\Delta y, z_0)$. The amount of shift $(\Delta x, \Delta y)$ is determined by the angle between the propagation vector $\mathbf{k}$ and the z-axis. Locally, the phase function of the optical field can be approximated by that of a plane wave (without loss of generality we assume the propagation vector lies in the xz-plane)

$$\theta(x,y,z) \approx -\mathbf{k}\cdot\mathbf{x} = -k\,[x\sin(\alpha) + z\cos(\alpha)], \tag{8}$$

where α is the angle between $\mathbf{k}$ and the z-axis, and k is the wavenumber. The derivative of the phase function with respect to x in the paraxial limit gives

$$\partial_x\theta \approx -k\alpha \approx -k\frac{\Delta x}{\Delta z}, \tag{9}$$

where we use Fig. 2 for the final expression. By solving Eq. (9) for Δx, we obtain an expression for the lateral drift. For the general paraxial case, the lateral drift $\Delta\mathbf{x}$ is given by

$$\Delta\mathbf{x} \approx \frac{-\nabla\theta\Delta z}{k} \quad \text{for} \quad k \ll |\nabla\theta|, \tag{10}$$

where $\nabla\theta$ represents the gradient of the phase function on the transverse plane. The approximation is valid in the paraxial limit.

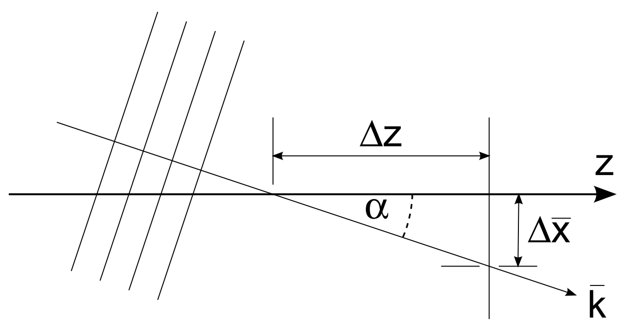

Figure 2: A wave propagating at some nonzero angle with respect to the z-axis produces a lateral drift of its transverse amplitude.

Since the complex amplitude as a whole experiences a lateral shift after a small propagation, the topological charge density will experience the same lateral shift after the same distance of propagation. We expand the resulting equation as a truncated Taylor series,

$$\begin{aligned} T(x,y,z+\Delta z) &= T(x-\Delta x, y-\Delta y, z) \\ &\approx T(x,y,z) - \Delta\mathbf{x}\cdot\nabla T(x,y,z). \end{aligned} \tag{11}$$

Substituting Eq. (10) into Eq. (11) and taking the limit $\Delta z \to 0$, one obtains,

$$\partial_z T(x,y,z) = \frac{\nabla\theta\cdot\nabla T(x,y,z)}{k}. \tag{12}$$

The gradient of the phase function that is produced by the distribution of vortices $\nabla\theta_S$ is given by the convolution between the topological charge density and the gradient of the phase function of a single optical vortex [16],

$$\nabla\theta_S = T(x,y,z)\otimes\nabla\phi, \tag{13}$$

where ϕ represents the phase function of a single positive canonical vortex and $\otimes$ denotes the convolution operation. Substituting Eq. (13) into Eq. (12), one obtains the expression for the drift term,

$$\partial_z T(x,y,z) = \frac{1}{k}\left[T(x,y,z)\otimes\nabla\phi\right]\cdot\nabla T(x,y,z). \tag{14}$$

The topological charge density $T(x,y,z)$ appears twice in this term, which means that the drift term is a nonlinear interaction term. It is also nonlocal because of the convolution process.

Note that the diffusion effect is not included in Eq. (14). In other words, we effectively set $\kappa_0 = 0$ to obtain the expression for the drift term. By adding the diffusion term, as it appears in the diffusion equation in Eq. (5), back into Eq. (14), one obtains a Fokker-Planck equation for the topological charge density, given by,

$$\partial_z T - \kappa_0 z\nabla^2 T - \frac{1}{k}(T\otimes\nabla\phi)\cdot\nabla T = 0. \tag{15}$$

We do not currently believe that Eq. (15), provides a complete description for the evolution of two-dimensional inhomogeneous topological charge distributions, because solutions of this equation do not exactly match results from numerical simulations (see Sec. 6). The missing part may be due to other drift mechanisms that are not currently understood.

4. Convolution

As noted in Eq. (13), the gradient of the phase function that is produced by the distribution of optical vortices is given by the convolution of the topological charge density and the gradient of the phase of a single vortex,

$$\begin{aligned}\nabla\theta_S &= T(x,y,z)\otimes\nabla\phi = \Xi\{T(x,y,z)\} \\ &= \iint_{-\infty}^{\infty} T(x-u,y-v,z)\nabla\phi(u,v)\,\mathrm{d}u\mathrm{d}v.\end{aligned} \tag{16}$$

Here we define this convolution process as an operator $\Xi\{\cdot\}$, operating on a topological charge density to produce a two-dimensional vector field. The gradient of the phase of a single positive canonical optical vortex is given by,

$$\nabla\phi(x,y) = \frac{x\hat{y}-y\hat{x}}{x^2+y^2}. \tag{17}$$

Since the topological charge density is real-valued, one can simplify the convolution expression. We define

$$F = (\hat{x}+i\hat{y})\cdot\nabla\theta_S, \tag{18}$$

so that the real (imaginary) part of F represents the x-component (y-component) of $\nabla\theta_S$. One can then recover $\nabla\theta_S$ through

$$\mathcal{R}e\{(\hat{x}-i\hat{y})F\} = \nabla\theta_S, \tag{19}$$

where $\mathcal{R}e\{\cdot\}$ takes the real part of the argument. Applying Eq. (18) to the gradient of the vortex phase function in Eq. (17), one obtains,

$$(\hat{x}+i\hat{y})\cdot\nabla\phi(x,y) = i\frac{(x+iy)}{x^2+y^2} = \frac{i}{x-iy}. \tag{20}$$

The expression for the convolution then simplifies to,

$$F = \iint_{-\infty}^{\infty}\frac{iT(x-u,y-v,z)}{u-iv}\,\mathrm{d}u\mathrm{d}v. \tag{21}$$

Now consider the case where $T = \cos(2\pi ax)$, with a representing the spatial frequency along x. We ignore the z-dependence for the moment, since it has no effect in the convolution process. Following the above procedure one finds,

$$\cos(2\pi ax)\otimes\nabla\phi = \frac{\hat{y}}{a}\sin(2\pi ax). \tag{22}$$

One can repeat this calculation for $\sin(2\pi ax)$, generalize these results for arbitrary directions and combine them to obtain the expression for all the Fourier components,

$$\begin{aligned}\Xi\{\exp(-i2\pi\mathbf{a}\cdot\mathbf{x})\} &= \exp(-i2\pi\mathbf{a}\cdot\mathbf{x})\otimes\nabla\phi \\ &= \frac{i(\hat{z}\times\mathbf{a})}{|\mathbf{a}|^2}\exp(-i2\pi\mathbf{a}\cdot\mathbf{x}),\end{aligned} \tag{23}$$

where $\mathbf{a} = a\hat{x}+b\hat{y}$.

So for a Fourier expansion of $T(x,y,z)$, given by,

$$T(x,y,z) = \iint_{-\infty}^{\infty}\tilde{T}(a,b,z)\exp(-i2\pi\mathbf{a}\cdot\mathbf{x})\ \mathrm{d}a\mathrm{d}b, \tag{24}$$

we have,

$$\Xi\{T\} = \iint_{-\infty}^{\infty}\tilde{T}(a,b,z)\frac{i(\hat{z}\times\mathbf{a})}{|\mathbf{a}|^2}\exp(-i2\pi\mathbf{a}\cdot\mathbf{x})\ \mathrm{d}a\mathrm{d}b, \tag{25}$$

for the general solution of the convolution process.

Since the convolution in the drift term always produces vector components orthogonal to the spatial dependence, while the gradient produces vector components parallel to the spatial dependence, it may appear to be impossible to get situations where the drift term is not zero. However, different Fourier components can interact with each other and thereby produce a nonzero drift term. To see this, we consider an initial topological charge density given by,

$$T(\mathbf{x},z=0) = \exp(-i2\pi\mathbf{a}_1\cdot\mathbf{x}) + \exp(-i2\pi\mathbf{a}_2\cdot\mathbf{x}), \tag{26}$$

ignoring for the moment the fact that T is supposed to be a real-valued function. This topological charge density produces a drift term given by,

$$\begin{aligned}(T\otimes\nabla\phi)\cdot\nabla T &= \frac{2\pi\,\hat{z}\cdot(\mathbf{a}_1\times\mathbf{a}_2)\left(|\mathbf{a}_1|^2-|\mathbf{a}_2|^2\right)}{|\mathbf{a}_1|^2|\mathbf{a}_2|^2} \\ &\quad\times\exp\left[-i2\pi(\mathbf{a}_1+\mathbf{a}_2)\cdot\mathbf{x}\right].\end{aligned} \tag{27}$$

It thus follows that, to have a nonzero drift term, the topological charge density must contain at least two Fourier components with different spatial frequencies that obey the following two requirements. Due to the factor $(|\mathbf{a}_1|^2 - |\mathbf{a}_2|^2)$ the spatial frequencies must have different magnitudes and due to the cross-product $(\mathbf{a}_1 \times \mathbf{a}_2)$ the spatial frequencies must also have different directions. The drift term will then introduce new Fourier components resulting from the product of the initial Fourier components.

The convolution operation has some interesting properties. If one computes the divergence of the resulting vector field one finds that $\nabla \cdot \Xi\{T\} = 0$. This is related to the fact that phase singularities don't survive a Laplacian operation. According to its definition $\Xi\{T\} = \nabla\theta_S$, where θ_S is the singular part of the phase function, it follows that $\nabla \cdot \Xi\{T\} = \nabla^2\theta_S = 0$. The fact that the divergence of $\Xi\{T\}$ vanishes, means that the drift term can be expressed as

$$(T \otimes \nabla\phi) \cdot \nabla T = \nabla \cdot [T\,(T \otimes \nabla\phi)] = \nabla \cdot (T\Xi\{T\})\,. \qquad (28)$$

Using this information one can then obtain a partial expression for the transverse topological charge current in Eq. (4), given by,

$$\mathbf{J}_T = -\kappa_0 z \nabla T - \frac{1}{k} T\Xi\{T\}. \qquad (29)$$

Moreover, we find that the curl of the resulting vector field gives $\nabla \times \Xi\{T\} = 2\pi T\hat{z}$. This is related to the index integral for phase singularities, which leads, via the Stokes theorem, to the fact that $\nabla \times \nabla\theta = \nabla \times \nabla\theta_S = 2\pi T\hat{z}$. In this case it is only the singular part of the phase function that can make a nonzero contribution to the right-hand side.

5. Analytical solutions

The simplest form that the initial topological charge density must have to produce a nonzero drift term is given by,

$$T(x, y, z = 0) = F_1 \cos(2\pi a_0 x) + F_2 \cos(2\pi b_0 y)\,, \qquad (30)$$

for any $a_0 \neq b_0$, where F_1 and F_2 are constant values. New Fourier components proportional to $\cos[2\pi(a_0 x + b_0 y)]$ and $\cos[2\pi(a_0 x - b_0 y)]$, are expected to be generated by the drift term. This is a prediction that can be tested with the aid of numerical simulations. For this purpose we need theoretical curves against which the numerical result can be compared.

We use a perturbative approach to solve Eq. (15), where we insert a small coupling constant for the nonlinear term in terms of which we then expand the solution. At the end the coupling constant is removed by setting it equal to 1. The reason why this approach produce solutions that agrees to some extent with the numerical solutions is because the contributions generated by the nonlinear term start out being small. The analytical solution thus found is

$$\begin{aligned} T(x, y, z) &= f_1(z)\cos(2\pi a_0 x) + f_2(z)\cos(2\pi b_0 y) \\ &\quad + f_3(z)\{\cos[2\pi(a_0 x + b_0 y)] \\ &\quad - \cos[2\pi(a_0 x - b_0 y)]\} \end{aligned} \qquad (31)$$

where

$$\begin{aligned} f_1(z) &= F_1 \exp(-2\pi^2\kappa_0 a_0^2 z^2) \\ &\quad \times \left\{1 - \frac{F_2^2 a_0^2(a_0^2 - b_0^2)}{4\kappa_0 b_0^4(a_0^2 + b_0^2)}\left[1 - \exp(-4\pi^2\kappa_0 b_0^2 z^2)\right]\right\} \\ f_2(z) &= F_2 \exp(-2\pi^2\kappa_0 b_0^2 z^2) \\ &\quad \times \left\{1 + \frac{F_1^2 b_0^2(a_0^2 - b_0^2)}{4\kappa_0 a_0^4(a_0^2 + b_0^2)}\left[1 - \exp(-4\pi^2\kappa_0 a_0^2 z^2)\right]\right\} \\ f_3(z) &= \frac{\pi F_1 F_2(a_0^2 - b_0^2)}{a_0 b_0} z \exp\left[-2\pi^2\kappa_0(a_0^2 + b_0^2)z^2\right] \end{aligned} \qquad (32)$$

This predicted evolution can now be compared with numerical simulations.

Note that, substituting $z = 0$ in Eq. (32), we get $f_3(0) = 0$. Hence, the mixed term is not present in the initial vortex distribution. Yet, due to the nonlinearity of the interaction term in Eq. (15), the mixed spatial frequency term is generated during the propagation of the optical field. The appearance of a new spatial frequency that was not present in the initial distribution is a clear indication that the dynamics which govern the evolution of the vortex distribution is nonlinear. So, although free-space propagation is a linear process acting on the optical field, the evolution of vortex distribution that is embedded in that optical field is nonlinear. This is not really as incredible as it may sound, because the vortex distribution is obtained by extracting the phase of the optical field, which requires a nonlinear calculation. The nonlinearity therefore enters as a result of the particular quantity that is being considered.

6. Numerical simulation

Numerical simulations are used to test the predicted nonlinear evolution of Eq. (15). The propagation of optical beams that contain specific distributions of optical vortices are simulated with a numerical beam propagation algorithm. The input to the simulations are sampled complex-valued functions, consisting of 512×512 pixels. Each sampled function represents a beam cross-section in the plane directly behind a phase-only diffractive optical element or a spatial light modulator that is used to produce the optical vortex distribution. The complex-valued functions are produced with periodic boundary conditions so that opposite edges of the function match each other continuously. In this way one avoids aliasing effects and the resulting optical field does not expand during propagation.

The numerical procedure then propagates the initial function through free-space over progressively larger distances. The propagation distance is incremented in steps to determine the effect of the propagation distance on the evolution of the topological charge distribution. For each step the topological charge density is obtained by computing the location and topological charge for every vortex in the beam. Such a topological charge density is a two-dimensional sampled function that is zero everywhere except at the locations of the vortices, where it is equal to either 1 or -1, indicating the topological charge of the

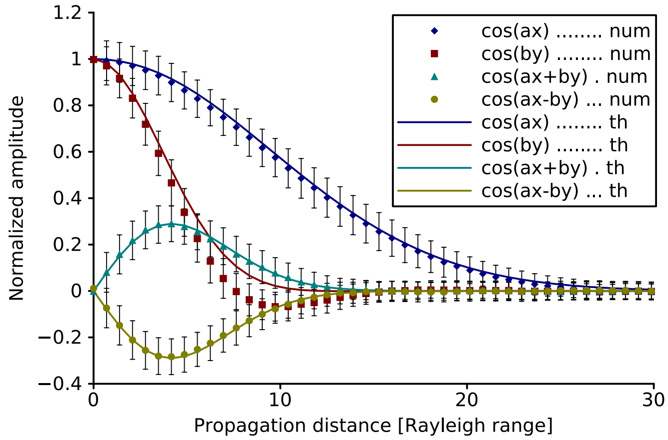

Figure 3: (color online) Decay curves of four different Fourier components as a function of the propagation distance, obtained from the numerical simulations of two-dimensional initial topological charge densities. The four Fourier components are those associated with $\cos(ax)$, $\cos(by)$, $\cos(ax+by)$ and $\cos(ax-by)$, where $a = 2\pi a_0$ and $b = 2\pi b_0$. The discrete points with the error bars represent results obtained from the numerical simulations and the solid curves are the theoretical predictions.

vortex. The Fourier transform of this topological charge density is used to extract the amplitudes of the pertinent Fourier components of the topological charge density.

For the comparison with the predicted theoretical curves we perform numerical simulations of the propagation of several different quasi-random vortex fields, each containing 820 vortices arranged to produce initial topological charge densities given by Eq. (30), with a_0 and b_0 chosen to give exactly 2 and 4 periods, respectively, inside the 512×512 window. The initial amplitudes F_1 and F_2 are both set equal to the maximum value that the initial vortex density allows. However, due to the randomness of the topological charge distribution the exact amplitude that is produced is not known *a priori*. We used a wavelength of 1.5 pixels, which is small enough to avoid the limitations that exist for topological charge densities [17].

During the simulated propagation we compute the topological charge density at regular intervals along the propagation direction and extract the Fourier components associated with $\cos(2\pi a_0 x)$, $\cos(2\pi b_0 y)$, $\cos[2\pi(a_0 x + b_0 y)]$ and $\cos[2\pi(a_0 x - b_0 y)]$ from these topological charge densities. The resulting curves of the amplitudes of these components are shown in Fig. 3, as a function of propagation distance. The propagation distance is measured in terms of the Rayleigh range $z_R = \pi L_c^2/\lambda$. The discrete points in Fig. 3 are the results from the numerical simulations and the error bars represent their standard deviations. The solid curves are the predictions obtained from Eqs. (31) and (32), which were fitted to the numerical data by matching the amplitude and position of the peak of $f_3(z)$ to compensate for the uncertainties in the amplitude and in the value of κ_0.

7. Discussion

Focusing on the numerical curves in Fig. 3, we see that the two initial Fourier component decay to zero at different rates. The one with the higher frequency component decays quicker than the lower frequency component, which is consistent with the diffusive behaviour found before [14]. We also see that two new Fourier components are generated as predicted, which confirms that the evolution of the topological charge distribution is nonlinear. Starting from zero, both of these new Fourier component reach nonzero values that are several times the standard deviation, after which they decay away to zero again. We also notice that the higher frequency component becomes negative with a magnitude slightly larger than one standard deviation before it decays to zero. This is also an indication of nonlinear evolution. These results therefore confirm that a nonlinear process is at work in the evolution of the topological charge distribution, as predicted by the drift term in Eq. (15).

After fitting the analytical curves to match the peak of the new Fourier components, one finds that all the curves match the numerical data remarkably well, accept for the initial Fourier component with the higher spatial frequency. Even in this case the discrepancy only lies in the region where the numerical data becomes negative. This discrepancy points to some additional drift mechanism(s) that has not yet been identified. On the other hand, the overall agreement that is seen is interpreted as confirmation that the phase drift mechanism, together with the previously reported diffusion mechanism [14], gives a good partial description of the evolution of a two-dimensional stochastic topological charge density.

8. Conclusion

We identified and formulated the phenomenon of phase drift in two-dimensional stochastic topological charge densities. Together with the previously identified diffusion effect, the phase drift effect is described by a differential equation for the topological charge density. Containing both a drift term and a diffusion term, this differential equation has the form of a Fokker-Planck equation. However, due to a slight discrepancy between numerical simulation results and the theoretical curves, this Fokker-Planck equation is not yet considered to be a complete description for the evolution of two-dimensional stochastic topological charge densities. Further investigations are needed.

The drift term is a nonlinear term, which accounts for part of the nonlinear evolution that is observed for two-dimensional stochastic topological charge densities. This nonlinear evolution is seen in the dynamics of the topological charge density during free-space propagation, which is in itself a linear process described by Maxwell's equations. The emergence of nonlinear dynamics from a fundamentally linear process can be seen as a result of the fact that the quantities under consideration have changed from being the full electromagnetic field to the distribution of phase singularities in the field. To extract the latter from the former, one requires a nonlinear operation. In this sense it is not surprising to see nonlinear dynamics. On the other hand, it also implies that the dynamics of optical vortex distributions cannot be directly derived from Maxwell's equations. The dynamics of optical vortex distributions is therefore an emergent phenomenon in the context of complexity theory.

References

[1] M. V. Berry, J. Phys. A: Math. Gen. 11 (1978) 27–37.
[2] N. B. Baranova, A. V. Mamaev, N. Pilipetsky, V. V. Shkunov, B. Y. Zel'dovich, J. Opt. Soc. Am. 73 (1983) 525–528.
[3] J. F. Nye, M. V. Berry, Proc. R. Soc. Lond. A 336 (1974) 165–190.
[4] P. Cullet, L. Gil, F. Rocca, Opt. Commun. 73 (1989) 403–408.
[5] N. Shvartsman, I. Freund, Phys. Rev. Lett. 72 (1994) 1008–1011.
[6] I. Freund, N. Shvartsman, V. Freilikher, Opt. Commun. 101 (1993) 247–264.
[7] I. Freund, J. Opt. Soc. Am. A 11 (1994) 1644–1652.
[8] M. V. Berry, M. R. Dennis, Proc. R. Soc. Lond. A 456 (2000) 2059–2079.
[9] W. Wang, S. G. Hanson, Y. Miyamoto, M. Takeda, Phys. Rev. Lett. 94 (2005) 103902.
[10] K. O'Holleran, M. R. Dennis, F. Flossmann, M. J. Padgett, Phys. Rev. Lett. 100 (2008).
[11] K. O'Holleran, M. R. Dennis, M. J. Padgett, Phys. Rev. Lett. 102 (2009).
[12] M. Chen, F. S. Roux, J. Opt. Soc. Am. A 27 (2010) 2138–2143.
[13] M. Chen, C. Dainty, F. S. Roux, Phys. Rev. A 84 (2011) 023846.
[14] F. S. Roux, Opt. Commun. 283 (2010) 4855–4858.
[15] F. S. Roux, J. Opt. Soc. Am. A 28 (2011) 621–626.
[16] F. S. Roux, J. Opt. Soc. Am. A 11 (1994) 2236–2243.
[17] F. S. Roux, Opt. Commun. 223 (2003) 31–37.