One- and two-dimensional topological charge distributions in stochastic optical fields

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our future through science

Statistical approach

It is not possible to formulate a general theory that can predict vortex trajectories $\mathbf{x}_n(z)$ from arbitrary initial vortex parameters

Reason: the vortex degrees of freedom are inseparable from other degrees of freedom in optical beams

However:

- ▷ Vortex dynamics may be predictable in a statistical sense
- ▷ Quantities would be defined in terms of probability distributions
- ▷ Justification: the other degrees of freedom average out
- ▷ Different perspective in terms of the kind of questions that are addressed

Definitions

Vortex number density: Number of vortices per cross-section area.
$\rightarrow$ function of transverse coordinates (x, y) that can changes as a function of propagation distance z

- $\triangleright$ Positive vortex density $n_p(x, y, z) \geq 0$
- $\triangleright$ Negative vortex density $n_n(x, y, z) \geq 0$
- $\triangleright$ Combined vortex density
  $V(x, y, z) = n_p(x, y, z) + n_n(x, y, z) \geq 0$
- $\triangleright$ Topological charge density
  $T(x, y, z) = n_p(x, y, z) - n_n(x, y, z)$

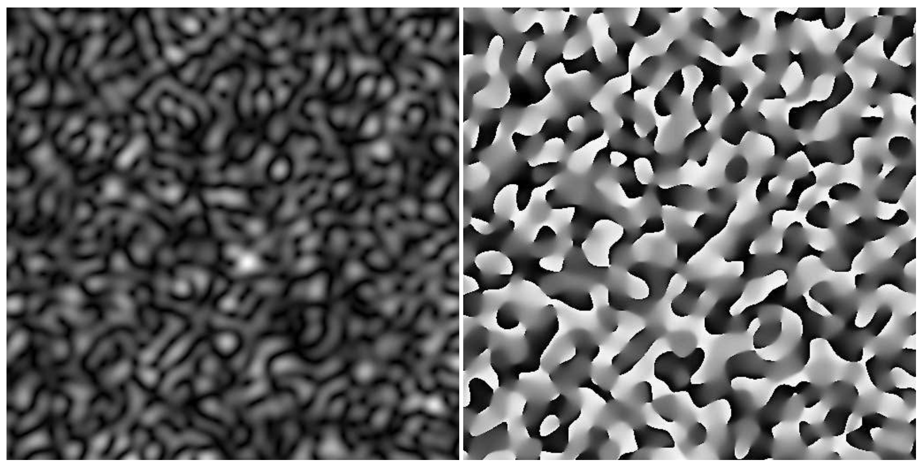
Speckle fields

Speckle field contains a random vortex field in equilibrium

- ▷ Globally: neutral topological charge
  ($\Leftrightarrow$ adjacent topological charges are anti-correlated)
- ▷ Annihilation rate = creation rate ($\Rightarrow$ equilibrium!)
- ▷ Equilibrium vortex density is determined by the properties of the speckle field[a]

$$V_{eq} = -\frac{\mathcal{C}''_{\mathbf{x}=0}}{4\pi} = \frac{A_c}{2}$$

A_c — coherence area
$\mathcal{C}$ — autocorrelation function



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[a] MV Berry, *J. Phys. A: Math. Gen.* **11**, 27-37 (1978);

N Shvartsman, I Freund, *Phys. Rev. Lett.* **72**, 1008-1011 (1994).

Topological charge density

Analytic calculation[a]

$$T_A = \frac{1}{A} \int_A \delta(\psi_r)\delta(\psi_i) \left(\partial_x \psi_r \partial_y \psi_i - \partial_x \psi_i \partial_y \psi_r\right) \, \mathrm{d}x\mathrm{d}y$$

$$T(\mathbf{x}) = \int \frac{\exp\left(-\mathbf{Q}^\dagger \mathbf{M}^{-1} \mathbf{Q}\right)}{\pi^3 \det(\mathbf{M})} (q_3 q_6 - q_5 q_4) \, \mathrm{d}^4 q \Bigg|_{q_1=q_2=0}$$

with

$$\mathbf{M} = \begin{bmatrix} \langle \psi\psi^* \rangle & \langle \psi_x \psi^* \rangle & \langle \psi_y \psi^* \rangle \\ \langle \psi\psi_x^* \rangle & \langle \psi_x \psi_x^* \rangle & \langle \psi_y \psi_x^* \rangle \\ \langle \psi\psi_y^* \rangle & \langle \psi_x \psi_y^* \rangle & \langle \psi_y \psi_y^* \rangle \end{bmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{bmatrix} q_1 + i q_2 \\ q_3 + i q_4 \\ q_5 + i q_6 \end{bmatrix}$$

$$T(\mathbf{x}) = \frac{i\left(\langle \psi_y \psi_x^* \rangle - \langle \psi_x \psi_y^* \rangle\right)}{2\pi \langle \psi\psi^* \rangle} + \frac{i\left(\langle \psi\psi_y^* \rangle \langle \psi_x \psi^* \rangle - \langle \psi_y \psi^* \rangle \langle \psi\psi_x^* \rangle\right)}{2\pi \langle \psi\psi^* \rangle^2}$$

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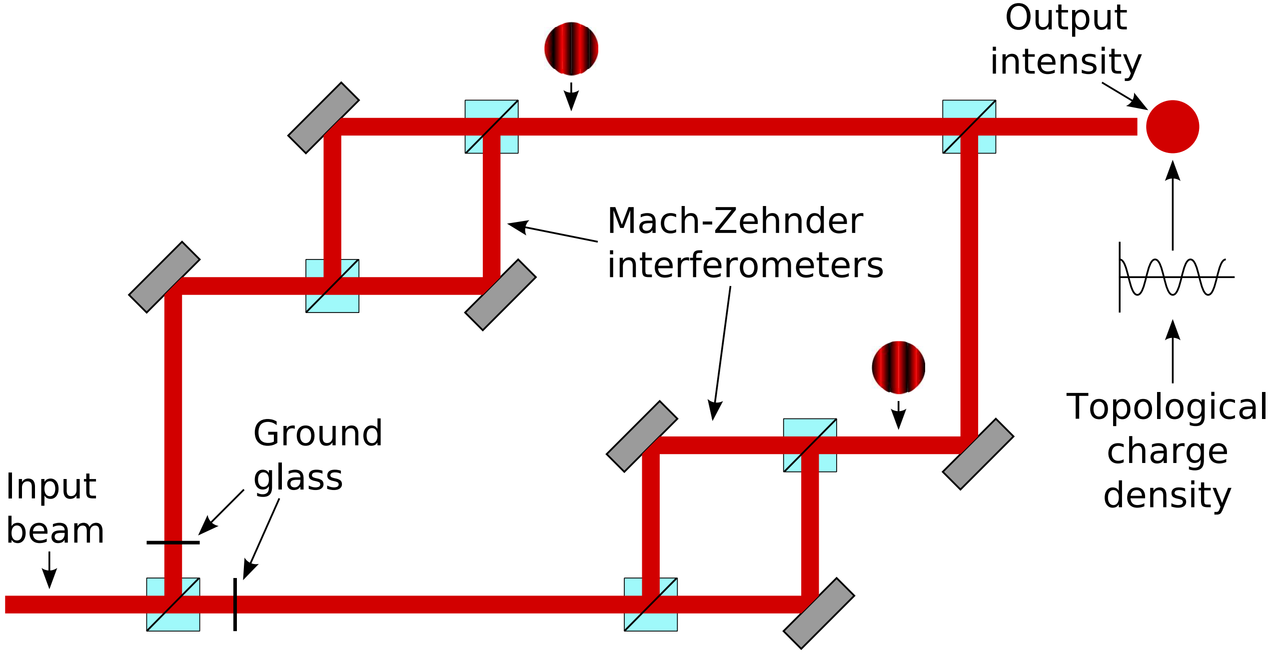
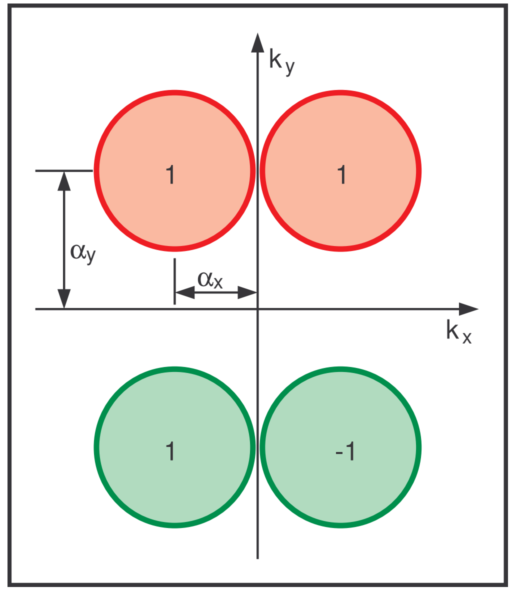
[a] MV Berry and MR Dennis, *Proc. R. Soc. London A* **456**, 2059-2079 (2000);

FS Roux, *J. Opt. Soc. Am. A* **28**, 621-626 (2011).

1D inhomogeneous fields

Experimental setup:





$$\psi_{in} = \tilde{\psi}_1 \sin(\alpha_x x) \exp(-i\alpha_y y) + \tilde{\psi}_2 \cos(\alpha_x x) \exp(i\alpha_y y)$$

Numerical simulation:

Beam propagation $\rightarrow$ extract vortex distribution

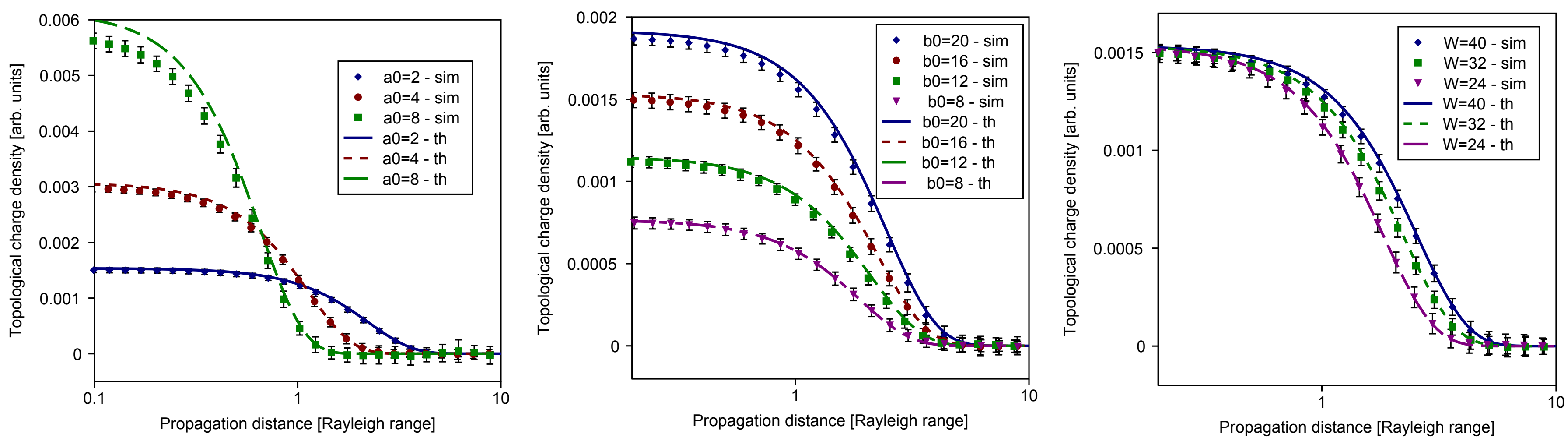
1D topological charge density

Analytical result:

$$T(x,z) = \frac{\alpha_x \alpha_y}{\pi} \sin(2\alpha_x x) \exp\left(-\frac{1}{2}\lambda^2 \alpha_x^2 W^2 z^2\right)$$

Definition: $\alpha_x = 2\pi a_0$ and $\alpha_y = 2\pi b_0$

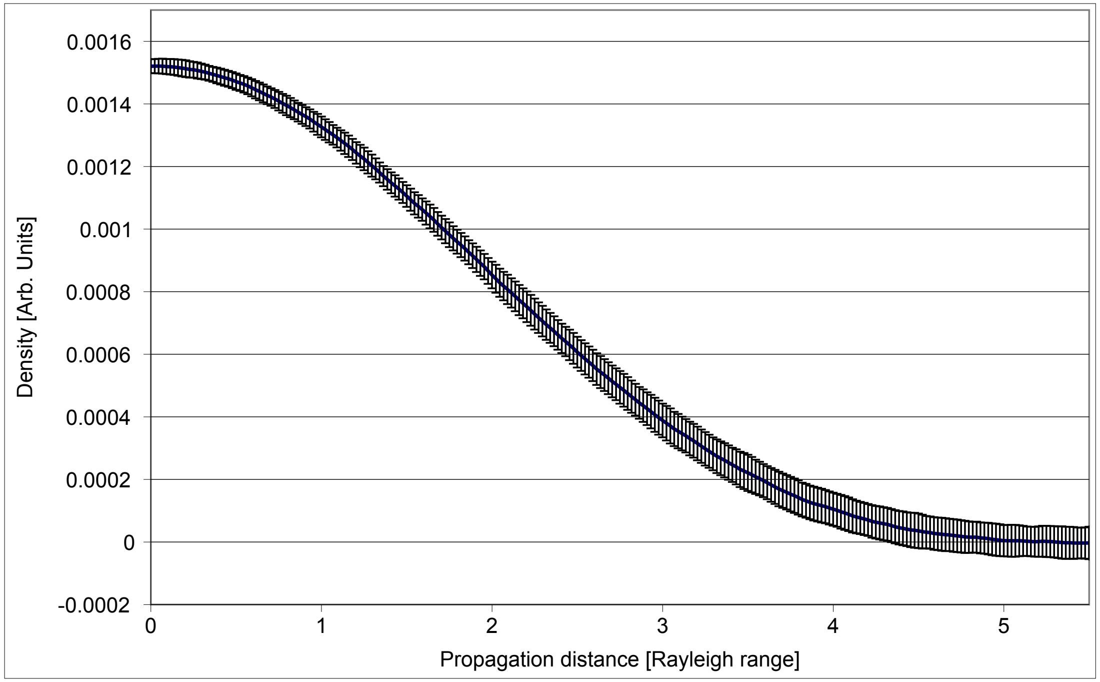
Comparison with numerical results: $(a_0 = 2, b_0 = 16, W = 32)$



1D topological charge density

Evolusion of topological charge density with input produced by direct phase modulation (SLM or DOE)

Numerical results:[a]



$$\partial_z T - z\kappa_0 \nabla^2 T = 0 \qquad \kappa_0 = \frac{\lambda^2}{\pi d^2} \qquad d = \text{coherence length}$$

[a] F.S. Roux, *Opt. Commun.* **283**, 4855-4858 (2010).

1D dynamics

Analytic result

$$T(x,z) = \frac{\alpha_x \alpha_y}{\pi} \sin(2\alpha_x x) \exp\left(-\frac{1}{2}\lambda^2 \alpha_x^2 W^2 z^2\right)$$

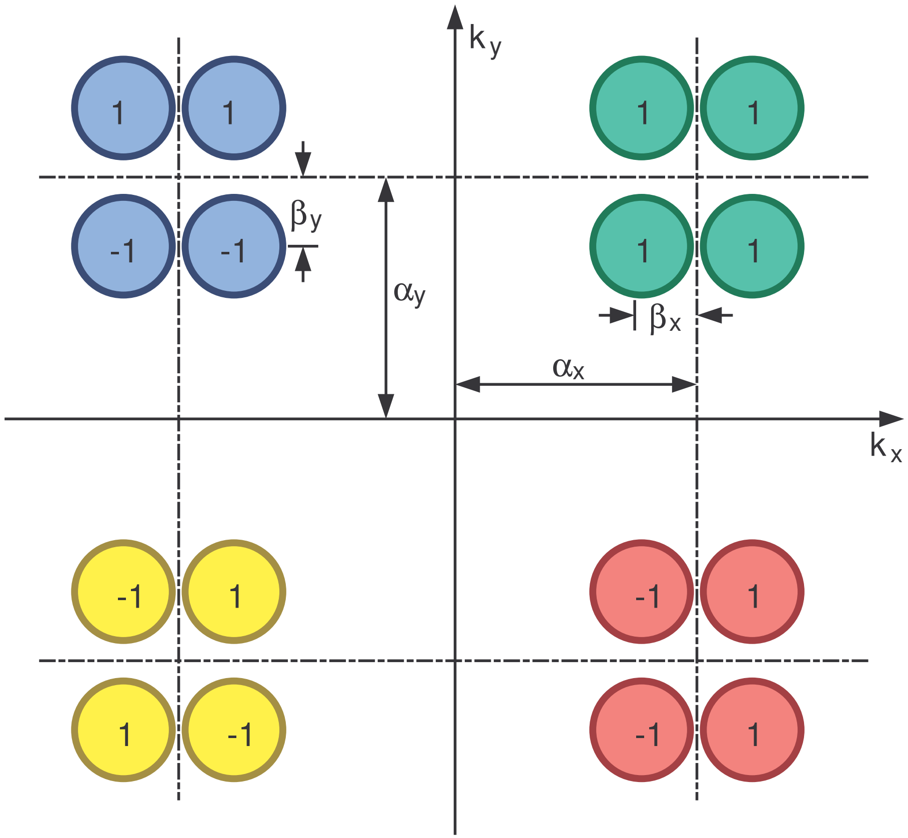
is a solution of: $\partial_z T - z\kappa_0 \nabla^2 T = 0$,

with $\kappa_0 = \frac{1}{4}\lambda^2 W^2 \quad \Rightarrow \quad d^2 = A_c = \frac{1}{\pi W^2}$

which is consistent with the definitions of the equilibrium vortex charge

2D inhomogeneous fields

Experimental setup is a generalization of the 1D case:



$$\psi_{in} = \tilde{\psi}_1 \cos(\beta_x x) \cos(\beta_y y) \exp[-i(\alpha_x x + \alpha_y y)]$$

$$+ \tilde{\psi}_2 \cos(\beta_x x) \sin(\beta_y y) \exp[i(\alpha_x x - \alpha_y y)]$$

$$+ \tilde{\psi}_3 \sin(\beta_x x) \cos(\beta_y y) \exp[-i(\alpha_x x - \alpha_y y)]$$

$$+ \tilde{\psi}_4 \sin(\beta_x x) \sin(\beta_y y) \exp[i(\alpha_x x + \alpha_y y)]$$

2D topological charge density

Analytical result:

$$T(\mathbf{x}) = \frac{f_1(z)\sin(2\beta_x x) + f_2(z)\sin(2\beta_y y) + f_3(z)\cos(2\beta_x x)\cos(2\beta_y y)}{\pi[1 + f_4(z)\sin(2\beta_x x)\sin(2\beta_y y)]^2}$$

where

$$\begin{aligned}
f_1(z) &= \frac{1}{2}\alpha_x\beta_y \exp[-z^2\eta(\beta_x^2 + 2\beta_y^2)]\sin(2zK_y)\sin(zK_x) \\
&\quad - \alpha_y\beta_x \exp(-\eta\beta_x^2 z^2)\cos(zK_x)
\end{aligned}$$

$$\begin{aligned}
f_2(z) &= -\frac{1}{2}\alpha_y\beta_x \exp[-z^2\eta(\beta_y^2 + 2\beta_x^2)]\sin(2zK_x)\sin(zK_y) \\
&\quad + \alpha_x\beta_y \exp(-\eta\beta_y^2 z^2)\cos(zK_y)
\end{aligned}$$

$$\begin{aligned}
f_3(z) &= \exp[-z^2\eta(\beta_x^2 + \beta_y^2)][\alpha_y\beta_x\sin(zK_x)\cos(zK_y) \\
&\quad - \alpha_x\beta_y\sin(zK_y)\cos(zK_x)]
\end{aligned}$$

$$f_4(z) = \exp[-z^2\eta(\beta_x^2 + \beta_y^2)]\sin(zK_y)\sin(zK_x)$$

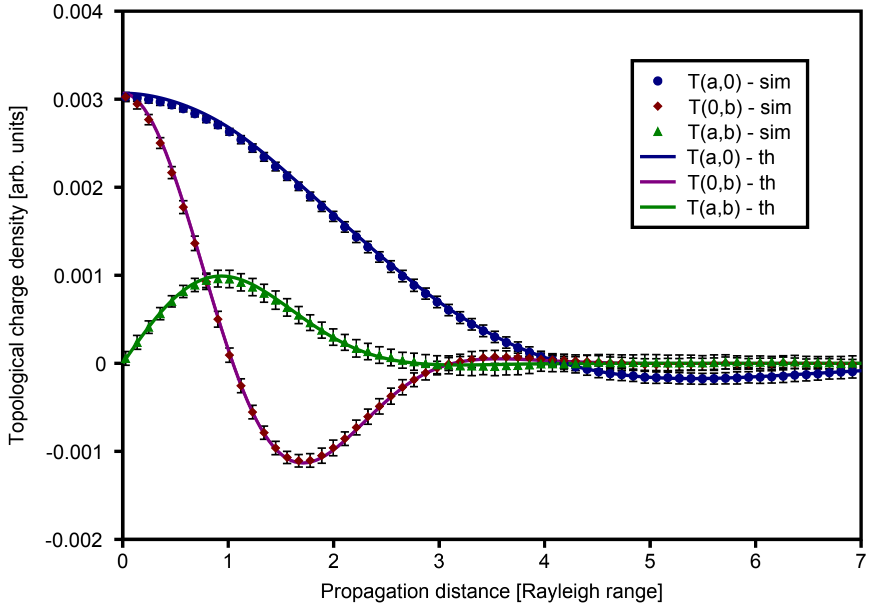
$$K_x = \frac{\lambda\alpha_x\beta_x}{\pi} \qquad K_y = \frac{\lambda\alpha_y\beta_y}{\pi} \qquad \eta = \frac{\lambda^2 W^2}{2}$$

Comparison with numerical results

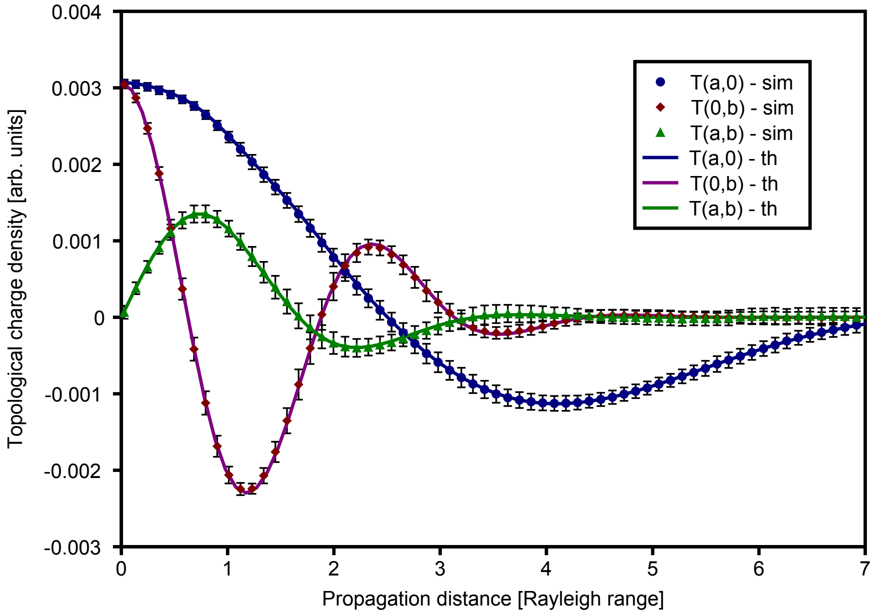
Definitions: $\alpha_x = 2\pi a_x$, $\alpha_y = 2\pi a_y$, $\beta_x = 2\pi b_x$ and $\beta_y = 2\pi b_y$

Parameters: $a_x = 2, a_y = 4, b_x = 16, b_y = 32$

$W = 32$



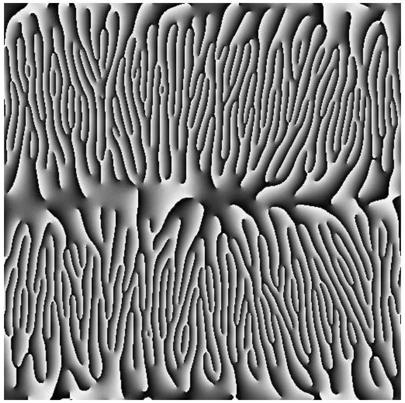
$W = 16$

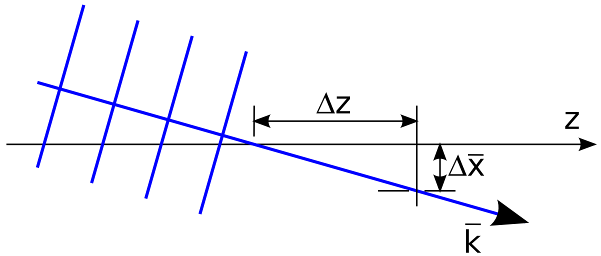


Topological charge evolusion

(Load tld.mpeg)

Phase drift


(a)



(b)

Topological charge $\rightarrow$ phase slope $\rightarrow$ sideways drift

$$\Delta \mathbf{x} \approx \frac{-\nabla\theta \Delta z}{k} \quad \text{for} \quad k \gg |\nabla\theta|$$

for $\Delta z \rightarrow 0$: $$\partial_z T(\mathbf{x}, z) = \frac{\nabla\theta \cdot \nabla T(\mathbf{x}, z)}{k}$$

Gradient of the phase function: $\nabla\theta = T(\mathbf{x}, z) \star \nabla\phi$

Drift term: $$\partial_z T(\mathbf{x}, z) = \frac{1}{k} \left[T(\mathbf{x}, z) \star \nabla\phi \right] \cdot \nabla T(\mathbf{x}, z)$$

where $\star =$ convolution and $$\nabla\phi(x, y) = \frac{y\hat{x} - x\hat{y}}{x^2 + y^2}$$

Summary

▷ Using combination of speckle fields one can produce inhomogeneous vortex distributions that allow both analytical calculations and numerical simulations

▷ One-dimensional topological charge density:

— Gaussian decay obeys (modified) diffusion equation

— Diffusion parameter is related to coherence area

▷ Two-dimensional topological charge density:

— The same diffusion behaviour

— Additional nonlinear behaviour may be explained by drift mechanisms