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COMPARISON OF CLASSICAL AND MODERN THEORIES OF LONGITUDINAL WAVE PROPAGATION IN ELASTIC RODS

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Abstract

A unified approach to derivation of different families of differential equations describing the longitudinal vibration of elastic rods and based on the Hamilton variational principle is outlined. The simplest model of longitudinal vibration of the rods does not take into consideration its lateral motion and is described in terms of the wave equation. The more elaborated models proposed by Rayleigh, Love, Bishop, Mindlin-Herrmann in which the lateral effects play an important role are also considered. The principles of construction of the multimode theories, corresponding equations and orthogonality conditions are considered. Dispersion curves, representing the eigenvalues versus real and imaginary values of the wave number, of these models are compared with the exact dispersion curves of an isotropic cylinder and conclusions on accuracy of the models are formulated.

1 Introduction

In what follows, the wave displacements in the rod are described in accordance with the assumptions made in various vibration theories. Hamilton's variational principle is then used to derive the equation or system of equations of motion corresponding to each approach. The method of finding the analytical solution is based on the separation of variables, the investigation of the eigenfunctions from the Sturm-Liouville problem, proof of two kinds of the eigenfunction orthogonality conditions by using the equations of the Sturm-Liouville problem. In the next stage the solution is assumed to be in the form of a Fourier series and is substituted into the Lagrangian which contains the Euler-Lagrange differential equation. The solution of the resulting differential equation is used together with the norms corresponding to the above orthogonalities to construct the Green functions.

2 Classical Theory: Wave Equation

In this case the longitudinal displacement is assumed constant in all points along the cross section of the rod and is expressed as $u = u(x,t)$. The general or compact form of the equation of motion is:

$$\frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial \dot{u}}\right)+\frac{\partial}{\partial x}\left(\frac{\partial \Lambda}{\partial u'}\right)-\frac{\partial \Lambda}{\partial u}=0 \tag{1}$$

with the natural boundary conditions $\left.\frac{\partial \Lambda}{\partial u'}\right|_{x=0,l}=0$ or $\left.u(x,t)\right|_{x=0,l}=0$, where

$$\Lambda=\Lambda(\dot{u},u',u)=\frac{1}{2}\left[A(x)\left(\rho(x)\dot{u}^2-E(x)u'^2\right)\right]+A(x)F(x,t)u \tag{2}$$

is the Lagrangian density, in which $A(x)$ is the cross-sectional area of the rod, $\rho(x)$ is the mass density of the rod $E(x)$ is the Young modulus of elasticity and $F(x,t)$ is the applied external force.
Substituting (2) into (1) we obtain the explicit form of the equation (1):

$$A(x)\rho(x)\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial x}\left(A(x)E(x)\frac{\partial u}{\partial x}\right)-A(x)F(x,t)=0 \tag{3}$$

The eigenfunctions follow from the corresponding Sturm-Liouville problem and satisfy two orthogonality conditions:

$$\int_0^l A(x)\rho(x)X_n(x)X_m(x)\,dx=0 \quad \text{and} \quad \int_0^l A(x)E(x)X_n'(x)X_m'(x)\,dx=0 \quad \text{for} \quad m\neq n \tag{4}$$

where $\|X_n\|_1^2=\int_0^l A\rho X_x^2dx$ and $\|X_n\|_2^2=\int_0^l AEX_n'^2dx$, and $X_n(x)$, $n=1,2,...$ are the eigenfunctions corresponding to the eigenvalues $\Omega_{1,n}=\frac{\|X_n\|_2}{\|X_n\|_1}$.

The solution of the problem is given by the following expression:

$$u(x,t)=\int_0^l g(\xi)\frac{\partial G(x,\xi,t)}{\partial t}d\xi+\int_0^l h(\xi)G(x,\xi,t)\,d\xi+\frac{1}{\rho}\int_0^t\int_0^l F(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau \tag{5}$$

where $G_1(x,\xi,t)=\sum_{n=1}^{\infty}\frac{X_n(x)X_n(\xi)\sin\Omega_{1,n}t}{\Omega_{1,n}\|X_n\|_1^2}$ is the Green function.

3 Rayleigh-Love Theory

The effects of the lateral displacement of the rod are taken into consideration in the kinetic energy by introducing the Poisson ratio η and the components of the displacement vector are[1]:

$$u=u(x,t), \qquad v=v(x,y)=-\eta y u_x', \qquad w=w(x,z)=-\eta z u_x' \tag{6}$$

and the equation of the motion in compact form is given as:

$$\left(\frac{\partial \Lambda}{\partial \dot{u}}\right)+\frac{\partial}{\partial x}\left(\frac{\partial \Lambda}{\partial u_x'}\right)-\frac{\partial^2}{\partial x\partial t}\left(\frac{\partial \Lambda}{\partial \dot{u}_x'}\right)-\frac{\partial \Lambda}{\partial u}=0 \tag{7}$$

with the associated boundary conditions $\left.\frac{\partial \Lambda}{\partial u'_x}-\frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial \dot{u}'_x}\right)\right|_{x=0,l}=0$ or $u(x,t)\big|_{x=0,l}=0$, where

$$\Lambda=\Lambda\left(u,\dot{u},u'_x,\dot{u}'_x\right)=\frac{1}{2}\rho(x)\left[A(x)\dot{u}^2+\eta(x)I_p(x)\dot{u}'^2_x-A(x)E(x)u'^2_x\right]+\int_0^l A(x)F(x,t)udx \tag{8}$$

is the Lagrangian density in which: I_p is the polar moment of inertia ($I_p=I_2=\int_s r^2ds=\pi\frac{R^4}{2}$ for the circular cross-section). By substituting (8) into (7) we obtain the Rayleigh-Love equation for the vibrating rod:

$$\underbrace{\rho(x)A(x)\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial x}\left(E(x)A(x)\frac{\partial u}{\partial x}\right)}_{\text{second order wave equation}}-\underbrace{\frac{\partial}{\partial x}\left(\rho(x)\eta^2(x)I_p(x)\frac{\partial^3 u}{\partial x\partial t^2}\right)}_{\text{Rayleigh-Love correction}}+A(x)F(x,t)=0 \tag{9}$$

The eigenfunctions $X_n(x),\ n=1,2,...$ are determined by solving the Sturm-Liouville problem corresponding to (9). These eigenfunctions satisfy the two orthogonality conditions:

$$\int_0^l\rho\left[AX_n(x)X_m(x)+\eta^2I_pX'_n(x)X'_m(x)\right]dx=0 \text{ and } \int_0^l EAX'_n(x)X'_m(x)dx=0 \text{ for } m\neq n \tag{10}$$

The solution of the problem is given by:

$$u(x,t)=\int_0^l A\left[g(\xi)\frac{\partial G_2(x,\xi,t)}{\partial t}+h(\xi)G_2(x,\xi,t)\right]d\xi+\int_0^l\eta^2 I_p\left[g'(\xi)\frac{\partial^2 G_2(x,\xi,t)}{\partial\xi\partial t}+h'(\xi)\frac{\partial G_2(x,\xi,t)}{\partial\xi}\right]d\xi+$$
$$+\frac{A}{\rho}\int_0^l\int_0^t F(\xi,\tau)G_2(x,\xi,t-\tau)d\tau d\xi \tag{11}$$

where $G_2(x,\xi,t)=\sum_{n=1}^{\infty}\frac{X_n(x)X_n(\xi)\sin\Omega_{2,n}t}{\Omega_{2,n}\left\|X_n\right\|_1^2}$ is the Green function and $\Omega_{2,n}=\frac{\left\|X_n\right\|_2}{\left\|X_n\right\|_1}$ are the eigenvalues associated with the above eigenfunctions.

4 Rayleigh-Bishop Theory

Certain assumptions (longitudinal and transverse displacements inside the excited rod) in the previous theory are kept and the vector displacement is the same. But in 1952, in order to improve the Rayleigh-Love theory, Bishop showed the contribution of shear stiffness accompanying the transverse deformation while calculating the strain energy[1, 2].

The general form of the equation of motion is:

$$\frac{\partial}{\partial t}\left(\frac{\partial\Lambda}{\partial\dot{u}}\right)+\frac{\partial}{\partial x}\left(\frac{\partial\Lambda}{\partial u'_x}\right)-\frac{\partial^2}{\partial x\partial t}\left(\frac{\partial\Lambda}{\partial\dot{u}'_x}\right)-\frac{\partial^2}{\partial x^2}\left(\frac{\partial\Lambda}{\partial u''_{xx}}\right)-\frac{\partial\Lambda}{\partial u}=0 \tag{12}$$

with the natural associated boundary conditions $\left.\frac{\partial\Lambda}{\partial u'_x}-\frac{\partial}{\partial t}\left(\frac{\partial\Lambda}{\partial\dot{u}'_x}\right)-\frac{\partial}{\partial x}\left(\frac{\partial\Lambda}{\partial u''_{xx}}\right)\right|_{x=0,l}=0$, and $u'_x(x,t)\big|_{x=0,l}=0$, or $u(x,t)\big|_{x=0,l}=0$, and $u''_{xx}(x,t)\big|_{x=0,l}=0$, where

$$\Lambda=\Lambda\left(u,\dot{u},u'_x,\dot{u}'_x,u''_{xx}\right)=\frac{1}{2}\rho(x)\left[A(x)\dot{u}^2+\eta(x)I_p(x)\dot{u}'^2_x\right]$$
$$-\frac{1}{2}\left[A(x)E(x)u'^2_x+\mu(x)I_p(x)u''^2_{xx}\right]+\int_0^l A(x)F(x,t)udx \tag{13}$$

is the Lagrangian density of the vibrating rod. By substituting (13) into (12), we obtain the explicit form of the equation:

$$\underbrace{\rho(x)A(x)\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial x}\left(E(x)A(x)\frac{\partial u}{\partial x}\right)}_{\text{second order wave equation}}-\underbrace{\frac{\partial}{\partial x}\left(\rho(x)\eta^2(x)I_p(x)\frac{\partial^3 u}{\partial x\partial t^2}\right)}_{\text{Rayleigh correction}}+\underbrace{\frac{\partial^2}{\partial x^2}\left(\mu(x)\eta^2(x)I_p(x)\frac{\partial^2 u}{\partial x^2}\right)}_{\text{Bishop correction}}-$$

$$-A(x)F(x,t)=0 \tag{14}$$

Investigation of the Sturm-Liouville problem corresponding to equation (14), shows that the resulting eigenfunctions $X_n(x),\ n=1,2,...$ satisfy two orthogonality conditions:

$$\int_0^l \rho\left[AX_n(x)X_m(x)+\eta^2 I_p X_n'(x)X_m'(x)\right]dx=0 \quad \text{and} \quad \int_0^l\left[EAX_n'(x)X_m'(x)+\mu\eta^2 I_p X_n''(x)X_m''(x)\right]dx=0 \tag{15}$$

The solution of equation (14) is given by:

$$u(x,t)=\int_0^l A\left[g(\xi)\frac{\partial G_3(x,\xi,t)}{\partial t}+h(\xi)G_3(x,\xi,t)\right]d\xi+\int_0^l \eta^2 I_p\left[g'(\xi)\frac{\partial^2 G_3(x,\xi,t)}{\partial\xi\partial t}+h'(\xi)\frac{\partial G_3(x,\xi,t)}{\partial \xi}\right]d\xi+$$

$$+\frac{A}{\rho}\int_0^l\int_0^t F(\xi,\tau)G_3(x,\xi,t-\tau)d\tau d\xi \tag{16}$$

where $G_3(x,\xi,t)=\sum_{n=1}^{\infty}\frac{X_n(x)X_n(\xi)\sin\Omega_{3,n}t}{\Omega_{3,n}\|X_n\|_1^2}$ is the Green Function, in which $\Omega_{3,n}=\frac{\|X_n\|_2}{\|X_n\|_1}$ are the natural eigenvalues corresponding to $X_n(x),\ n=1,2,...$.

5 Mindlin-Herrmann Theory

Despite the fact that has improved the previous theories. It is necessary to emphasise the lack of physical clarity in interpretation of certain high-order modes, mainly independent shear and radial motion. In order to address this insufficiency Mindlin and Herrmann take into account the independent shearing deformation, radial displacement and distributed stress along the transversal direction[3]. According to these new ideas the displacements are represented by two independent functions:

$$u=u(x,t)=\Phi_0(x,t), \qquad v=v(x,r,t)=r\cdot\Phi_1(x,t) \tag{17}$$

where r is the distance between the points along the lateral direction of the rod.

The compact form of the system of equations of motion is given as follows:

$$\frac{d}{dt}\left(\frac{\partial\Lambda}{\partial\dot{\Phi}_k}\right)+\frac{d}{dx}\left(\frac{\partial\Lambda}{\partial\Phi_k'}\right)-\frac{\partial\Lambda}{\partial\Phi_k}=0, \qquad (k=0,1) \tag{18}$$

With the corresponding set of natural boundary conditions $\Phi_0(x,t)\big|_{x=0,l}=0,\ \Phi_1(x,t)\big|_{x=0,l}=0$, or $\left.\frac{\partial\Lambda}{\partial\Phi_1'}\right|_{x=0,l}=0,\ \left.\frac{\partial\Lambda}{\partial\Phi_0'}\right|_{x=0,l}=0,$ where

$$\Lambda=\Lambda\left(\dot{\Phi}_0,\dot{\Phi}_1,\Phi_0',\Phi_1',\Phi_1\right)$$

$$=\frac{\rho}{2}\left(A\dot{\Phi}_0^2(x,t)+I_2\dot{\Phi}_1^2(x,t)\right)-\frac{1}{2}\left((\lambda+2\mu)\Phi_0'^2A+4\lambda\Phi_0'\Phi_1A+4(\lambda+\mu)\Phi_1^2A+I_2\mu\Phi_1'^2\right) \tag{19}$$

is the Lagrangian density of the rod, in which $\mu=\frac{E}{2(1+\eta)}$ and $\lambda=\frac{E\eta}{(1-2\eta)(1+\eta)}$, are Lame's constants. Substituting expression (19) into the system (18) leads to the explicit form of the system of equations:

$$\begin{cases} \rho A\partial_t^2\Phi_0 - (\lambda+2\mu)A\partial_x^2\Phi_0 - 2\lambda A\partial_x\Phi_1 = AF(x,t) \\ 2\lambda A\partial_x\Phi_0 + \rho I_p\partial_t^2\Phi_1 - \mu I_p\partial_x^2\Phi_1 + AS(\lambda+\mu)\Phi_1 = 0 \end{cases} \tag{20}$$

The couple of eigenfunctions found by investigating the Sturm-Liouville problem associated to the system (14), hold the following orthogonality properties for $m \neq n$ $\int_0^l \rho\left(A\Phi_{0,n}\Phi_{0,m} + I_p\Phi_{1,n}\Phi_{1,m}\right)dx = 0$ and

$\int_0^l \left\{4A(\lambda+\mu)\Phi_{1,n}\Phi_{1,m} + I_p\mu\Phi'_{1,n}\Phi'_{1,m} + A(\lambda+2\mu)\Phi'_{0,n}\Phi'_{0,m} + 2\lambda A\left(\Phi'_{0,n}\Phi_{1,m} + \Phi'_{0,m}\Phi_{1,n}\right)\right\}dx = 0$.

The solution of the system of equations (20) can be found:

$$\Phi_0(x,t) = \int_0^l Ag(\xi)\frac{\partial G_4(x,\xi,t)}{\partial t}d\xi + \int_0^l I_p\phi(\xi)\frac{\partial G_5(x,\xi,t)}{\partial t}d\xi +$$
$$+\int_0^l Ah(\xi)G_4(x,\xi,t)d\xi + \int_0^l I_p\varphi(\xi)G_5(x,\xi,t)d\xi + \frac{1}{\rho}\int_0^t\int_0^l F(\xi,\tau)G_4(x,\xi,t-\tau)d\tau d\xi$$
$$\Phi_1(x,t) = \int_0^l Ag(\xi)\frac{\partial G_6(x,\xi,t)}{\partial t}d\xi + \int_0^l I_p\phi(\xi)\frac{\partial G_7(x,\xi,t)}{\partial t}d\xi +$$
$$+\int_0^l Ah(\xi)G_6(x,\xi,t)d\xi + \int_0^l I_p\varphi(\xi)G_7(x,\xi,t)d\xi + \frac{1}{\rho}\int_0^t\int_0^l F(\xi,\tau)G_6(x,\varepsilon,t-\tau)d\tau d\xi \tag{21}$$

where $u(x,r,t)\big|_{t=0} = \phi(x)$ and $\dot{u}(x,r,t)\big|_{t=0} = \varphi(x)$ are initial transverse displacement and velocity,

$$G_4(x,\xi,t) = \sum_{n=1}^{\infty}\left(\frac{\Phi_{0,n}(x)\Phi_{0,n}(\xi)\sin\Omega_{4,n}t}{\Omega_{4,n}\left\|(\Phi_{0,n},\Phi_{1,n})\right\|_1^2}\right), G_5(x,\xi,t) = \sum_{n=1}^{\infty}\left(\frac{\Phi_{0,n}(x)\Phi_{1,n}(\xi)\sin\Omega_{4,n}t}{\Omega_{4,n}\left\|(\Phi_{0,n},\Phi_{1,n})\right\|_1^2}\right),$$
$$G_6(x,\xi,t) = \sum_{n=1}^{\infty}\left(\frac{\Phi_{1,n}(x)\Phi_{0,n}(\xi)\sin\Omega_n t}{\Omega_{4,n}\left\|(\Phi_{0,n},\Phi_{1,n})\right\|_1^2}\right), G_7(x,\xi,t) = \sum_{n=1}^{\infty}\left(\frac{\Phi_{1,n}(x)\Phi_{1,n}(\xi)\sin\Omega_{4,n}t}{\Omega_{4,n}\left\|(\Phi_{0,n},\Phi_{1,n})\right\|_1^2}\right),$$

are the Green functions, and $\Omega_{4,n} = \dfrac{\left\|(\Phi_{0,n},\Phi_{1,n}),(\Phi_{0,n},\Phi_{1,n})'\right\|_2}{\left\|(\Phi_{0,n}\Phi_{1,n})\right\|_1}$ are the eigenvalues.

6 Multimode Theories

A more accurate description of rod deformation can be obtained by increasing the number of possible deformation modes. The so called Mindlin-McNiven theory[4] is one such multimode model. Here we consider another multimode generalisation of the Mindlin-Herrmann model of longitudinal vibration of a rod with circular cross-section. Assume the axisymmetric case where the displacements are approximated as follows:

$$\begin{aligned} u &= u(x,r,t) = \Phi_0(x,t) + r^2\Phi_2(x,t) + ... + r^{2i}\Phi_{2i}(x,t); \quad i = 0,1,2,... \\ v &= v(x,r,t) = r\Phi_1(x,t) + r^3\Phi_3(x,t) + ... + r^{2j+1}\Phi_{2j+1}(x,t), \quad j = 0,1,2,... \end{aligned} \tag{22}$$

According to the choice of i and j we can obtain a higher or lower mode of vibration of the rod.

In the case $i=1$, $j=0$: $u = u(x,r,t) = \Phi_0(x,t) + r^2\Phi_2(x,t)$ and $v = v(x,r,t) = r\Phi_1(x,t)$ and the system of equations of motion in general form is

$$\frac{\partial}{\partial t}\left(\frac{\partial\Lambda}{\partial\dot{\Phi}_k}\right) + \frac{\partial}{\partial x}\left(\frac{\partial\Lambda}{\partial\Phi'_k}\right) - \frac{\partial\Lambda}{\partial\Phi_k} = 0, \qquad (k = 0,1,2) \tag{23}$$

with a set of natural boundary conditions $\Phi_k(x,t)\big|_{x=0,l}=0$ and $\left.\dfrac{\partial\Lambda}{\partial\Phi_k(x,t)}\right|_{x=0,l}=0$, where

$$\Lambda=\Lambda\left(\dot\Phi_0,\dot\Phi_1,\dot\Phi_2,\Phi_0',\Phi_1',\Phi_2',\Phi_1,\Phi_2\right)$$
$$=\frac{1}{2}\left[\left(A\dot\Phi_0^2+2I_2\dot\Phi_0\dot\Phi_2+I_2\dot\Phi_1^2+I_4\dot\Phi_2^2\right)-(\lambda+2\mu)A\Phi_0'^2+\mu I_2\Phi_1'^2+(\lambda+2\mu)I_4\Phi_2'^2+4\lambda A\Phi_0'\Phi_1\right]-$$
$$-\frac{1}{2}\left[4\mu I_2\Phi_1'\Phi_2+2(\lambda+2\mu)I_2\Phi_0'\Phi_2'+4\lambda I_2\Phi_2'\Phi_1+4(\lambda+\mu)A\Phi_1^2+4\mu I_2\Phi_2^2\right] \tag{24}$$

is the Lagrangian density of the rod and $I_4=\int_s r^4ds=\pi\dfrac{R^6}{3}$.

Substituting expression (24) into system (23) we obtain the explicit form of the system of equation of motion in operator form:

$$A\left[\rho\partial_t^2-(\lambda+2\mu)\partial_x^2\right]\Phi_0-\left[2\lambda A\partial_x\right]\Phi_1+I_2\left[\rho\partial_t^2-(\lambda+2\mu)\partial_x^2\right]\Phi_2=0$$
$$\left[2\lambda A\partial_x\right]\Phi_0+\left[I_2\left(\rho\partial_t^2-\mu\partial_x^2\right)+4A(\lambda+\mu)\right]\Phi_1+\left[2(\lambda-\mu)I_2\partial_x\right]\Phi_2=0$$
$$I_2\left[\rho\partial_t^2-(\lambda+2\mu)\partial_x^2\right]\Phi_0-\left[2(\lambda-\mu)I_2\partial_x\right]\Phi_1+\left[I_4\left(\rho\partial_t^2-(\lambda+2\mu)\partial_x^2\right)+4\mu I_2\right]\Phi_2=0 \tag{25}$$

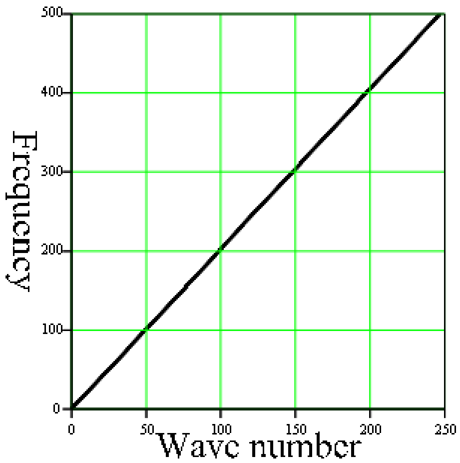
The orthogonality conditions are:

$$\int_0^l\left[A\Phi_{0,n}\Phi_{0,m}+I_p\left(\Phi_{0,n}\Phi_{2,m}+\Phi_{2,n}\Phi_{0,m}+\Phi_{1,n}\Phi_{1,m}\right)+I_4\Phi_{2,n}\Phi_{2,m}\right]dx=0,$$
$$\int_0^l\Big\{A\left[(\lambda+2\mu)\Phi_{0,n}'\Phi_{0,m}'+4(\lambda+\mu)\Phi_{1,n}\Phi_{1,m}+2\lambda\left(\Phi_{0,n}'\Phi_{1,m}+\Phi_{1,n}'\Phi_{0,m}\right)\right]$$
$$+I_p\Big[(\lambda+2\mu)\left(\Phi_{0,n}'\Phi_{2,m}'+\Phi_{2,n}'\Phi_{0,m}'\right)+2\lambda\left(\Phi_{1,n}\Phi_{2,m}'+\Phi_{1,m}\Phi_{2,n}'\right)$$
$$+\mu\left(\Phi_{1,n}'\Phi_{1,m}'+2\left(\Phi_{1,n}'\Phi_{2,m}+\Phi_{1,m}'\Phi_{2,n}\right)+4\Phi_{2,m}\Phi_{2,n}\right)\Big]+I_4(\lambda+2\mu)\Phi_{2,m}'\Phi_{2,n}'\Big\}dx=0$$

for $m\neq n$.

7 Comparison of Different Theories

We analyse different models of longitudinal vibration of rods by drawing their frequency spectra and comparing them with the frequency spectrum of the exact Pochhammer-Chree solution[5] of the axisymmetric problem of a cylindrical rod with free outer surface. To make this comparison we assume $u(x,t)=U\cdot e^{i(\omega t-kx)}$, $\Phi_k(x,r,t)=\Phi_k(r)\cdot e^{i(\omega t-kx)}$ and substitute these values in (3), (9),(14),(20), and (25). It is supposed in this case that all parameters of equations are constant (say, $A(x)=A=const$, etc.). In the classical case we obtain a single straight line $\omega(k)=\sqrt{E/\rho}\cdot k$. The frequency spectrum curve $\left(\omega\cdot R\cdot\sqrt{\rho/\mu}\right)$ versus of $(k\cdot R)$ of the Rayleigh-Love model is shown in Figure 1 for $k\cdot R\in(0,20]$, $\omega\cdot R\cdot\sqrt{\rho/\mu}\in(0,32]$, where R – radius of the outer cylindrical surface of the rod (all other figures are drawn in the same ranges). Figure 2 demonstrates the frequency spectrum of the Rayleigh-Bishop model.



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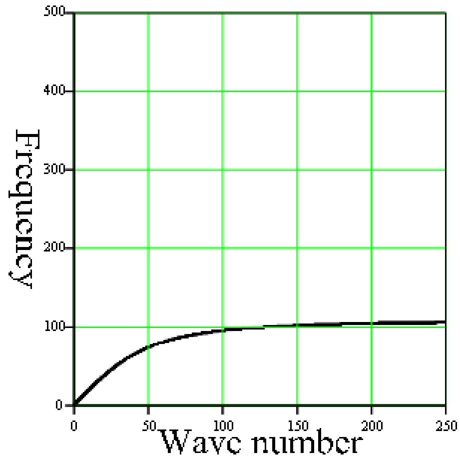

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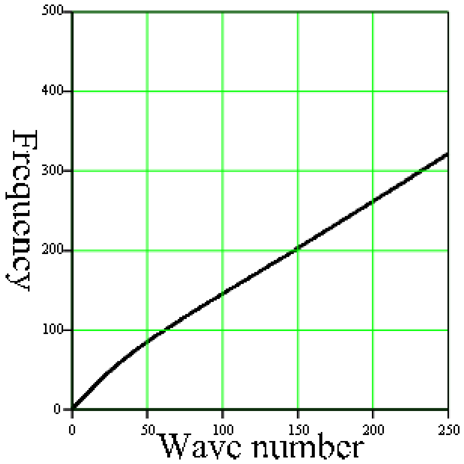
Figure 1. Rayleigh-Love model.

Figure 2. Rayleigh-Bishop model.

The frequency spectrum of the Mindlin-Herrmann model is shown in Figure 3. Figure 4 illustrates the multimode model with $i=1,\ j=0$.



RRBB

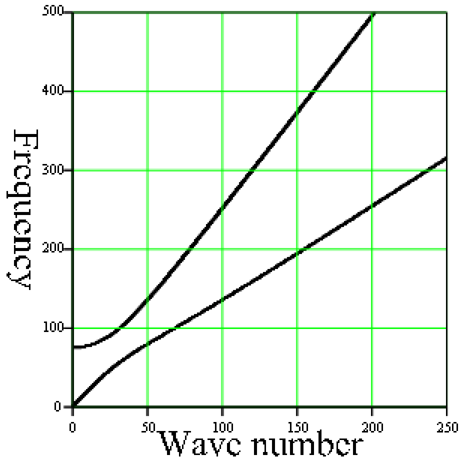


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Figure 3. Mindlin-Herrmann model

Figure 4. Multimode model ($i=1,\ j=0$)

Figure 5 illustrates the multimode model with $i=2,\ j=1$ and Figure 6 shows the exact Pochhammer-Chree model of the axisymmetric case and free cylindrical surface.

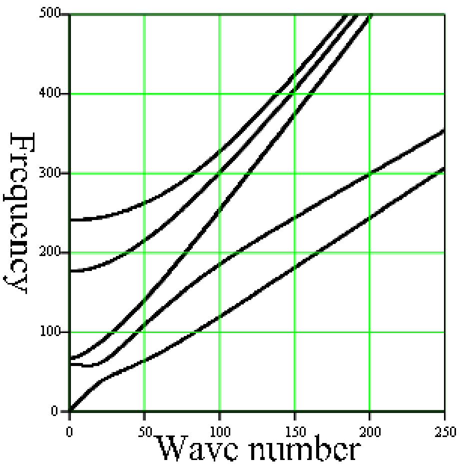

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Figure 5. Multimode model ($i = 2,\ j = 1$)

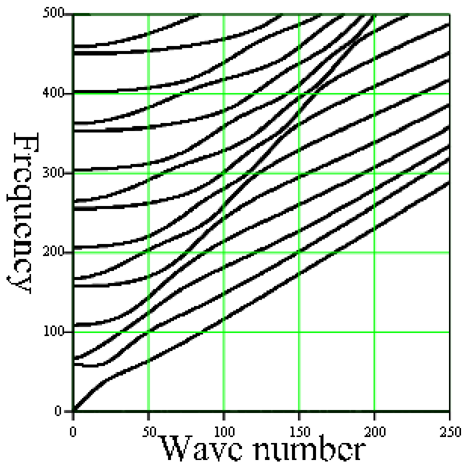

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Figure 6. Pochhammer –Chree model.

8 Conclusion

In the present paper we compared the classical, Rayleigh-Love, Rayleigh-Bishop, Mindlin-Herrmann, and multimode models of longitudinal vibration of rods with the exact Pochhammer-Chree solution for axisymmetric vibration of an isotropic cylinder with free outer surface. The classical, Rayleigh-Love, and Rayleigh-Bishop models approximately describe the first mode of the exact solution in a restricted "k - ω"- domain. The Rayleigh-Bishop approximation is more accurate, but the frequency spectrum asymptotically tends to the shear wave solution while the exact solution tends to the surface waves mode. It is explained by the hypothesis on one dimensionality in all the above approximate models. The Mindlin-Herrmann model also satisfies the plane cross-section hypothesis. Due to the fact that this model is described in terms of two independent functions the frequency spectrum contains two branches. In the multimode model we reject the hypothesis on plane cross-section and obtain more branches. The higher the order of the multimode approximation the broader is the *"k -* ω*"*- domain in which the effects of longitudinal vibrations of the rods could be analysed.

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