Optimal Placement of Range-only Beacons for Mobile Robot Localisation

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***Abstract*—The ability of an agent to self-localise is crucial to any autonomous task where mobility is required. A common set of techniques solving the localisation problem involve the deployment of active beacons or landmarks, which eliminate problems related to landmark detection and association. The use of beacons providing range-only estimates using time-of-flight measurements is one such approach. Here, range measurements are used in trilateration or range-only SLAM algorithms to provide an accurate measure of a robot's position. Unfortunately, the potential error in a position estimate is related to the relative geometry of the beacons, and poorly placed beacons can result in extremely inaccurate location estimates. This paper presents an optimisation technique for finding optimal beacon positions, so as to minimise the mean positional uncertainty in a given environment. Our work shows that this approach represents an improvement on previous approaches because the resultant uncertainty map can be used as a heuristic to improve path planning algorithms.**

I. Introduction

An accurate estimate of a robot's position is typically required prior to any meaningful interaction with an environment. As a result, methods with which a robot can be localised are of great interest to the robotics research community. Traditional approaches to localisation require a map, but in many environments this is unavailable. This has led to investigation into a number of alternative localisation techniques. These include the widely used global positioning system (GPS), simultaneous localisation and mapping (SLAM) approaches and inertial navigation systems. Unfortunately, SLAM approaches that do not make use of fiducials or distinct landmarks often lack the robust performance required in industrial settings, while the cost and drift associated with inertial navigation systems typically make them infeasible. Due to this, active landmarks, which eliminate common problems related to landmark detection and association, are often deployed in industrial work areas.

Beacons providing time-of-flight measurements are frequently used as active landmarks. In this case, the position of a receiver is estimated using measurements of the ranges to respective beacons. Approaches that allow for this include trilateration, multilateration and range-only SLAM. Trilateration is a method of determining the absolute or relative locations of points by measurement of distances, using the geometry of circles, spheres or triangles. Multilateration, also known as hyperbolic positioning, refers to the process of locating an object by accurately computing the time difference of arrival of signals emitted by three or more receivers. Range-only SLAM, discussed in detail in [1], [2] and [3], is a localisation approach that typically makes use of active beacons to acquire range measurements. Trilateration or multilateration is used to determine an initial estimate of the robot's position and a filtering algorithm that integrates range measurements from multiple beacons over time is then applied. Range measurements are typically noisy, especially in an indoor environment where interference and multipath affects signal quality, so this integration result tends to drift over time.

In all cases, the relative geometry of the beacons affects the accuracy of localisation. Thus, a strategy for the optimal placement of beacons is required. This paper presents an optimisation technique for beacon placement that minimises the mean positional uncertainty in an environment.

Optimal range-only beacon placement strategies have been applied previously. The authors of [4] developed a nonlinear mixed-integer programming model to minimise the number of beacons covering an environment, given the requirement that at least three beacons must be within a specified range of critical points in the design area. Unfortunately, this approach does not take positional uncertainty estimates into account and could result in beacon configurations with the potential for large positional errors in specific positions.

In [5], a diversified local search strategy to find the optimal number and position of beacons for a given environment was applied. The authors aimed to find an optimal solution for beacon positioning with respect to three conditions; a minimal number of beacons must be used, the percentage of the area covered by at least three beacons must be maximised, and the percentage of area covered by admissible geometric dilution of precision (GDOP) values must be maximised. GDOP is an uncertainty scaling parameter dependent on the relative geometry of beacons, which was originally developed for global positioning systems.

GDOP is defined as the square root of the sum of the variances in position error, divided by the average variance in beacon range measurements [6]. This measure has two primary flaws. Firstly, it fails to take off-diagonal covariance in position estimates into account, and assumes that beacon variance is the same for all beacons. The latter assumption is suitable for GPS, where satellites are so far away that they can be assumed to be equidistant from the receiver, but this does not necessarily hold for indoor beacons, where variances are typically highly dependent on range. Secondly, the exclusion of off-diagonal terms in the dilution of precision calculation fails to accurately measure the potential error in estimates for certain beacon configurations. The work presented here provides a more suitable metric for the positional uncertainty associated with a given beacon configuration, which addresses the failings of the GDOP metric.

The work of [5] makes use of a diversified search strategy, where movements out of local minima are automated. At present, our approach relies on manual intervention to avoid local minima. In addition, our method does not find the minimum number of beacons required to achieve a specified certainty over the intended environment. However, our approach does produce a resultant uncertainty map, which can be used as a heuristic to improve path planning. Our work also highlights the flaws of applying a standard least squares solution to estimate position using all detected beacons, which may result in a sub-optimal level of positional uncertainty.

This paper is organised as follows. In Section II, trilateration and an approach to finding an optimal set of beacon positions for a given environment is discussed. Section III provides a discussion on the primary benefit of this approach to practical situations, the use of the resultant uncertainty map as an aid in path planning. This is followed by a discussion on the relevance of the beacon positions if range-only SLAM is used in the environment. Experimental results showing uncertainty maps for beacons during the optimisation process are provided in Section IV, together with results of an A* planning algorithm using the uncertainty maps as a heuristic. Finally, conclusions and a discussion on future work are provided in Section V.

II. METHODOLOGY

The beacon placement strategy presented here involves the minimisation of the average uncertainty in a given environment. This requires knowledge of the effects of beacon noise on a position estimate. The following section provides trilateration equations which can be used to estimate receiver position when the locations of beacons are known. This is followed by a discussion on a suitable noise model for beacons, and a derivation of the effects of this uncertainty on a position estimate. Finally, we show how the uncertainty in a position measurement can be used to determine an optimal set of beacon locations.

A. Trilateration

Trilateration is a means of determining the absolute location of a point, using the geometry of circles or spheres. Assuming at least four range measurements are obtained from beacons at known locations, the location of a receiver in 3D space can be determined by finding the intersections of the four spheres. This process is illustrated for a single plane in 3D space in Fig. 1. The trilateration solution can be generalised to multiple beacons as follows. Assuming N beacons, with i denoting the i-th beacon in a set, the i-th range measurement for a given receiver position in 3D Cartesian space (x, y, z) is calculated as

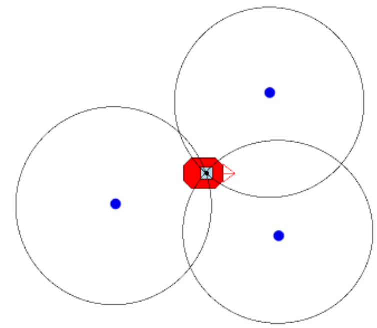


Fig. 1. A graphical illustration of the trilateration process in 2D.

$$r_i^2 = (x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 . \quad (1)$$

Here, (x_i, y_i, z_i) denotes the position of the i-th beacon in Cartesian space. Expanding (1), and substituting $t = x^2 + y^2 + z^2$ and $S_i = x_i^2 + y_i^2 + z_i^2$ provides

$$S_i - r_i^2 = 2xx_i + 2yy_i + 2zz_i - t. \quad (2)$$

Assuming N beacons, (2) can be formulated as a least squares problem, $\mathbf{Ax} = \mathbf{B}$,

$$\begin{bmatrix} 2x_1 & 2y_1 & 2z_1 & -1 \\ \vdots & \vdots & \vdots & \vdots \\ 2x_N & 2y_N & 2z_N & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} S_1 - r_1^2 \\ \vdots \\ S_N - r_N^2 \end{bmatrix} \quad (3)$$

and the receiver position is easily solved as $\mathbf{x} = \mathbf{A}^\dagger \mathbf{B}$, with $\mathbf{A}^\dagger$ the Moore-Penrose pseudo-inverse of $\mathbf{A}$. Hereafter, the left most matrix of (3) will be referred to as $\mathbf{A}$.

B. Beacon Noise Model

The optimisation approach to beacon placement presented here relies on the propagation of uncertainty in beacon measurements through the trilateration equations. However, before this can be completed, knowledge of the mean uncertainty for a given beacon measurement is required. Most range-only beacons in use rely on some form of time-of-flight measurement to estimate range. This approach suffers from attenuation effects and multi-path. As a result, errors in range estimate become more pronounced as the distance between beacon and receiver increases. Assuming each beacon estimate has zero-mean Gaussian noise, a beacon noise model that accounts for this has a standard deviation in error given by

$$\sigma = \begin{cases} \sigma_c r^2 & r_{min} \leq r \leq r_{max} \\ \infty & \text{otherwise} \end{cases} , \quad (4)$$

where σ_c is a tuning constant for specific beacons, and r is the distance between receiver and beacon. In practice, a suitable value for σ_c is obtained through experimentation and by characterising the error in a set of beacons. r_{min} and r_{max} denote the respective minimum and maximum ranges achievable by the beacons.

C. Uncertainty Propagation in Trilateration

The noise model of (4) is now used to obtain the uncertainty in a receiver position estimate by propagating the model variance σ^2 through the trilateration equations of (3).

Re-writing (3), to include Gaussian noise variables ϵ_i for the i-th range estimate provides

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \mathbf{A}^{\dagger} \begin{bmatrix} S_1 - (r_1 + \epsilon_1)^2 \\ \vdots \\ S_N - (r_N + \epsilon_N)^2 \end{bmatrix}. \quad (5)$$

Using Taylor series expansion to linearise the right hand side of (5) about the zero-mean Gaussian variables ϵ_i then results in

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \mathbf{A}^{\dagger} \begin{bmatrix} S_1 - {r_1}^2 - 2r_1\epsilon_1 \\ \vdots \\ S_N - {r_N}^2 - 2r_N\epsilon_N \end{bmatrix}. \quad (6)$$

Given a linear system $\mathbf{Y} = \mathbf{T}\mathbf{X}$, the transform of the covariance of Gaussian random variables passed through the system is given by $\mathrm{Cov}\left[\mathbf{Y}\right] = \mathbf{T}\,\mathrm{Cov}\left[\mathbf{X}\right]\mathbf{T}^{\mathrm{T}}$. Thus, the covariance for a given receiver position can be estimated from a diagonal matrix of beacon range variances using

$$\mathrm{Cov}\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \mathbf{T} \begin{bmatrix} \sigma_1^2 & \dots & 0 \\ \vdots & \ddots & 0 \\ 0 & 0 & \sigma_N^2 \end{bmatrix} \mathbf{T}^{\mathrm{T}}, \quad (7)$$

where

$$\mathbf{T} = \mathbf{A}^{\dagger} \begin{bmatrix} -2r_1 & \dots & 0 \\ \vdots & \ddots & 0 \\ 0 & 0 & -2r_N \end{bmatrix}. \quad (8)$$

At this point, it is important to highlight a crucial point of interest regarding the potential uncertainty in an estimated receiver position for a given set of beacons. The transform $\mathbf{T}$ is range dependent, which means that the uncertainty in receiver position is not only dependent on the relative beacon geometry, but also on the distance to beacons. Intuitively, it would seem that the addition of beacons to a given area will always improve a position estimate, but this is not always true. In many cases, *the addition of a beacon to the least squares problem will result in an increased uncertainty and a greater chance of error* in a position estimate.

In order to remedy this, we recommend that the uncertainties for all potential combinations of beacons are evaluated, and that the set of beacons providing the lowest positional uncertainty is selected. An even better approach towards minimising uncertainty for a given beacon configuration would be to use a weighted least squares position solution, but this is left as future work, and its exclusion does not affect the optimisation process described here.

D. Optimisation Process

Given the uncertainty estimate of (7), the optimisation process for finding the best beacon positions for a given environment can now be discussed. For a given position, only the upper left 3×3 elements of the covariance matrix in (7) are of interest, as they contain all positional uncertainty information. Ideally, these elements should be combined to form one quality metric relating to the potential error in a position estimate. We now describe a suitable metric, which relies on principal component analysis.

Principal component analysis (PCA), first described in [7], is a dimensionality reduction technique that transforms data into a coordinate system in which the largest variances in the data lie on coordinate axes termed principal components. Mathematically, the magnitudes of the principal components are the square roots of the eigenvalues of a covariance matrix. Thus, by calculating the eigenvalues of the upper left 3×3 elements in (7), the largest orthogonal variances for a given position estimate are obtained. A suitable error metric using these rotated variances is the square root of the sum of these variances:

$$\mathrm{e}_{x,y,z} = \sqrt{\sum_{k=1}^{3} \left(\mathrm{Eig}\left[\mathrm{Cov}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right] \right)}. \quad (9)$$

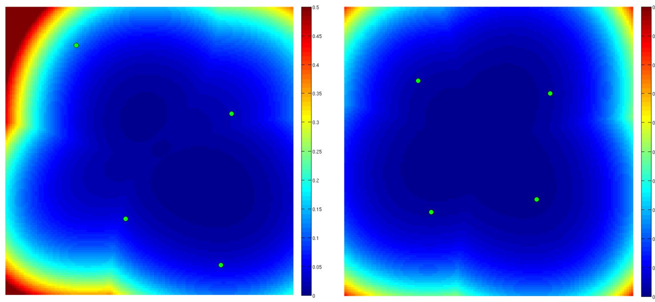
This can be considered as a measure of the Euclidean distance between the position estimate (x, y, z) and the largest orthogonal standard deviations in 3D space, and is an improvement on the GDOP metric as it considers both off-diagonal covariance terms and individual beacon noise.

The beacon positioning problem is now formulated as an optimisation problem using this error metric,

$$[\bar{x}_i, \bar{y}_i, \bar{z}_i] = \underset{\bar{x}_i, \bar{y}_i, \bar{z}_i}{\operatorname{argmin}} E\left[\mathrm{e}_{x,y,z}\right] \quad (10)$$

where $(x, y, z) \in S$, the bounds on a given environment, and $(\bar{x}_i, \bar{y}_i, \bar{z}_i)$ denote vectors containing the respective beacon coordinates. In other words, choose the set of beacon positions so as to minimise the mean, denoted $E\left[\,.\,\right]$, positional uncertainty for a given environment, S. Almost any nonlinear optimisation algorithm could be applied to solve this problem, but we selected the Nelder-Mead [8] downhill simplex method due to its applicability to solving highly non-linear optimisation problems. Initial beacon positions are selected at random or by a user, and the Nelder-Mead algorithm converges to a locally optimal solution. A suitable set of beacon positions is obtained by repeating the Nelder-Mead search a number of times.

In practice, obstacles such as walls do not allow for range measurements to be made at all locations. In order to account for this, our algorithm assumes that range measurements for respective beacons are only available if there is an unobstructed line-of-sight between the beacon and receiver position. The effects of multi-path are assumed to be dealt with by increased attenuation thresholds on the receiver, at the expense of additional beacons. In addition, we ensure that the minimum uncertainty is obtained for a given position and



(a) Iteration 1 (b) Iteration 1000

Fig. 2. Convergence of beacons to a minimum uncertainty map with no obstacles. Beacons are denoted by green circles, while the uncertainty map is a colour based representation of the uncertainty in an environment, with blue representing low levels of uncertainty, and red greater uncertainty.

beacon configuration by selecting the best set of beacons for the computation, as outlined in Section II-C.

Unfortunately, the optimisation process described here is extremely slow. In order to improve the speed of the process, a staged search for optimum beacon positions is used. Initially, the search for the best beacon locations occurs on an extremely course grid representation of the environment. The results of this search are then used to seed a search on a more densely populated grid, which allows for the refinement of beacon positions.

III. Discussion

A. Uncertainty as a Path Planning Heuristic

The optimisation process of Section II-D produces a set of beacon positions and an uncertainty map for the intended operational environment. This uncertainty map is of particular importance as it can be used as an aid in path planning. Ideally, a path between two points should be selected so that the uncertainty in position along the path is minimised. This would reduce the chances of negative effects due to potential localisation errors when a mobile robot is navigating. We now show how the uncertainty map can be used as a heuristic for the popular A* path planning algorithm.

The A* search algorithm, first described in [9], is a commonly used extension of the Dijkstra algorithm [10] that allows for the shortest route between two points or nodes on a graph to be found, provided that the two nodes are connected in some manner. Given a start location, the algorithm tests connected nodes in an attempt to find the lowest cost route to a goal. The cost at each node is evaluated as the sum of the total cost to reach the node and a heuristic measure indicating an estimated cost to reach the goal. A priority queue of the nodes to be traversed is maintained, and termed the open set. As the total cost of reaching the goal via a node decreases, the nodes priority increases. At each iteration of the algorithm, the lowest cost node is removed, and added to a closed set. This process continues until the lowest total cost path reaching a goal is obtained, at which point the desired route is found by working backwards from the goal node using information contained in the closed set.

Traditionally, the heuristic used by the A* algorithm for mobile robot path planning is the Euclidean distance between an accessed node and a goal. This results in the shortest route to a goal being determined. The cost of traversing to a node is typically the distance between nodes. However, by replacing this cost with the uncertainty metric at a node, and the heuristic with the maximum map uncertainty metric scaled by the distance between an accessed node and a goal, a path that minimises the potential for errors in localisation will be obtained. This path may not be ideal from the perspective of time taken to reach the goal however, and a better approach may be to replace the heuristic with the Euclidean distance to the goal. This ensures that the path to the goal is near optimal in terms of shortest distance travelled, but still takes potential uncertainty into account at each iteration.

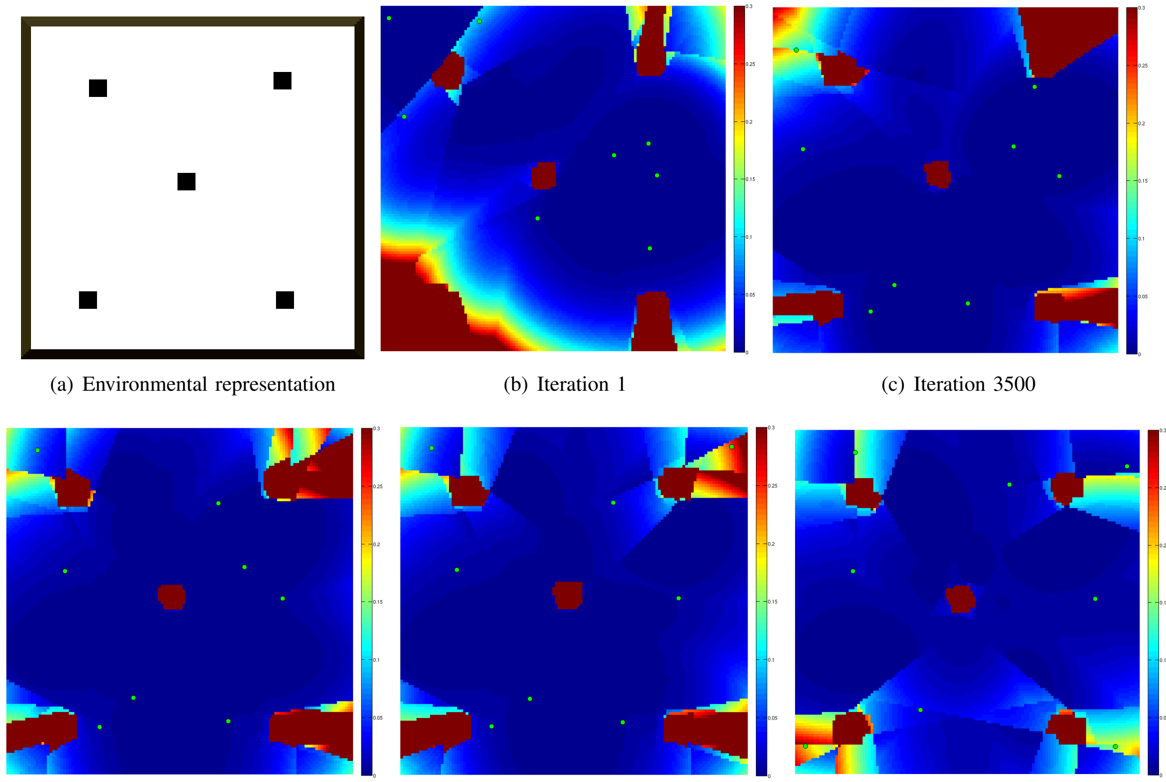
B. Applicability to Range-only SLAM

In many cases trilateration is not the only approach used to estimate receiver position, and a position fusion algorithm is applied. This is especially true for mobile robot localisation, where vehicle odometry is frequently used to improve a position estimate. This process is typically termed range-only SLAM. Care must be taken to discriminate between the true range-only SLAM problem and one where beacon positions are known a-priori.

In the latter case, localisation occurs in a global frame since the positions of beacons are known. Thus, the SLAM problem is reduced to one of localisation, and the combination of odometry and measurement information merely serves to improve the uncertainty in a position estimate. This is simply a case of sensor fusion, and as a result the beacon placement strategy discussed here is still optimal.

However, for true SLAM, where the positions of beacons are not known, and need to be estimated continuously, the beacon placement strategy described here *may not be optimal*. In the case of true SLAM, beacon and robot positions are estimated in a relative frame. As a result, uncertainty in position is subject to drift and bounded at infinity. Positional uncertainty will continue to increase over time, unless loop closure is applied. Loop closure refers to the adjustment of position estimates and uncertainties when a robot returns to a previously visited location, and is an essential component in SLAM algorithms if positional uncertainty is to be bound.

The importance of loop closure cannot be overstated. The authors of [11] showed that errors in robot mapping can be reduced dramatically by using a navigation strategy that alternates between exploring and returning to known points for loop closure. The significance of loop closure in a SLAM algorithm implies that a better beacon configuration would be one where the uncertainty in a position estimate is minimised at positions where loop closure is likely to occur, such as at frequently traversed intersections. However, the uncertainty map produced by the optimisation algorithm described here could be used to plan returns to areas of low localisation uncertainty, which would allow for more reliable loop closure.



(a) Environmental representation (b) Iteration 1 (c) Iteration 3500

(d) Iteration 3713 (e) Iteration 4035 (f) Iteration 10000

Fig. 3. Convergence of beacons to a minimum uncertainty map. Beacons are denoted by green circles, while the uncertainty map is a colour based representation of the uncertainty in an environment, with blue representing low levels of uncertainty, and red greater uncertainty.

IV. Experimental Results

A. Beacon Positioning

Experimental results of our beacon placement strategy are now presented. We restrict our analysis to two dimensions, so as to aid understanding, but it is important to note that the algorithms presented here are applicable to three dimensional problems. Fig. 2 shows a graphical representation of a sample uncertainty map at various stages during an optimisation process for a set of beacons with $\sigma_c = 0.006$, when no obstacles restrict the beacon line-of-sight. The error metric has units of metres, and the environment considered is square with an area of $100\,\mathrm{m}^2$. Beacons are denoted by green circles. Initially, 4 beacons were placed at random, and the average error metric was quite large. However, this decreased rapidly, with beacon locations found after 1000 iterations resulting in a mean positional uncertainty of $0.44\,\mathrm{m}$, approximately 14% lower than the $0.51\,\mathrm{m}$ initially obtained.

Fig. 3 shows a graphical representation of a sample uncertainty map at various stages during an optimisation process for a set of beacons with $\sigma_c = 0.006$, when 5 obstacles restrict beacon line-of-sights. Once more, the environment considered is square, with an area of $100\,\mathrm{m}^2$, and beacons are denoted by green circles. In this experiment, 8 beacons were placed at random, as a greater number of beacons are required to counter the obstructions due to obstacles. As before, the mean uncertainty decreased rapidly during the position optimisation. However, the Nelder-Mead optimisation algorithm occasionally found local minima, which required that the algorithm be restarted with beacon positions adjusted by the user. This occurred twice, between iterations 1 and 3500, and again between iterations 3713 and 4035.

Initially, the mean uncertainty in position was $0.90\,\mathrm{m}$, which was reduced to $0.40\,\mathrm{m}$, an impressive reduction of approximately 65%. It is important to note that this average excludes the uncertainty within obstacles. Obstacles have extremely large positional uncertainty because no knowledge of positions within obstacles can be found. The final beacon positions provide lower uncertainty behind obstacles, but this comes at the expense of greater uncertainty in other areas. In practice, this drop in certainty can be limited by using the median of the uncertainty map instead of the mean as a cost function, since this will result in outliers affecting the optimisation problem to a lesser extent.

B. Path Planning

Fig. 4 shows the results of three variations of an A* path planning algorithm, adapted to find a path between two goals, using a traditional shortest distance metric, a combination of a minimum uncertainty cost and maximum uncertainty heuristic, and a mixture of a minimum uncertainty cost with a Euclidean distance heuristic. These three cost functions were discussed in Section III-A.

As expected, using the shortest distance metric in an A* algorithm resulted in the shortest distance path between the start and target nodes being found. A metric based solely on the uncertainty in the environment found a path between the goals that, although minimising uncertainty, is extremely lengthy. As predicted, the third path, found using an uncertainty cost to transition between points, but a Euclidean distance heuristic to the goal, resulted in a much shorter path, but allowed for variation about the path which reduced the uncertainty in each A* iteration. In practice, we would recommend the use of this planning strategy.

V. Conclusion

This work has presented an optimisation process for the placement of range-only localisation beacons. Beacons are placed so as to minimise the mean uncertainty or variance in a position measurement over a given environment, for a given beacon's Gaussian noise model. This paper has shown how the uncertainty in beacon range measurements propagates through trilateration equations, and affects the receiver's position estimate.

Of crucial importance is the fact that adding beacons does not always improve the uncertainty in a position estimate, since errors in range measurements are dependent on the distance to beacons. We have provided a method for selecting the best set of beacons to estimate position at a given point.

The optimisation algorithm approach to beacon placement is able to improve the mean uncertainty in an environment and is suitable for use in selecting beacon locations for an industrial application. Unfortunately, the optimisation problem presented here is extremely computationally intensive and as a result quite slow. The use of alternative optimisation algorithms and modification of the error metric so as to improve speed is a matter of future work.

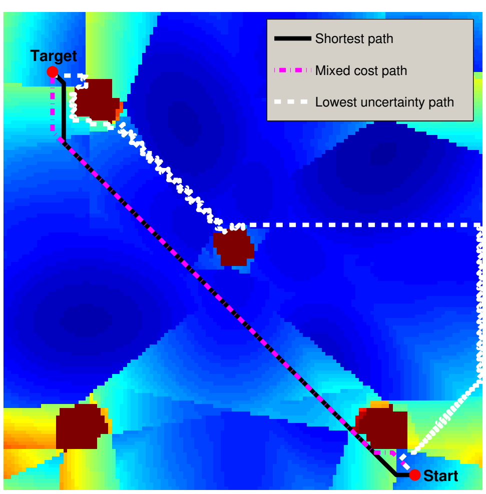


Fig. 4. A set of sample paths between two points in an uncertainty map. The path planned using a mixed metric produced the most feasible path, while still allowing for navigation with reduced uncertainty.

While the beacon configurations found using the optimisation approach discussed here may not be optimal if a true range-only SLAM algorithm is employed, this is not really of consequence. The design process described here implicitly requires that knowledge of beacon positions is known a-priori, and a range-only SLAM algorithm is not really suited to solving a localisation problem in this form. A much better approach is that of sensor fusion, in which case the method presented here is suitable for selecting beacon positions.

The primary benefit of the uncertainty-based beacon placement strategy presented here is the use of the uncertainty map for mobile robot planning. This paper has shown how the localisation uncertainty in an area can be used to plan paths that minimise the chances of positional errors affecting navigation.

The approach presented here has only considered the optimal placement of a constant number of beacons. Future work involves the adaption of the approach to select the optimal number and position of beacons in order to provide a desired level of certainty in a given environment. The increased weighting of the uncertainty of areas where accurate localisation is essential is another planned addition to the beacon placement strategy presented here, as this will allow even greater control over the localisation errors in a given environment.

Acknowledgment

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