Investigation of a local spatial spectrum of Bessel light beams

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Abstract
In this paper we consider the angular spectrum of an apertured Bessel beam when the aperture is circular and shifted laterally with respect to the optical axis. Since the perturbation of the resulting angular spectrum is due to a spatially localized Bessel field, we refer to such an angular spectrum as the local spatial spectrum. We show that the local spatial spectrum of the Bessel beam may differ considerably from the angular spectrum of an unapertured Bessel beam, and find that the ring-shaped angular spectrum is converted into two arcs whose width in the azimuthal direction monotonically decreases as the shift of the circular aperture increases. This phenomenon is predicted theoretically and confirmed experimentally.

Key words: Bessel beams, spatial spectrum, Fourier transformation, axicon, annular and circular aperture.

1 Introduction

An important feature of light beams is the structure of their spatial (or angular) spectrum. For example, knowledge of the angular spectrum of the field allows for propagation through linear and non-linear optical systems since the angular spectrum is nothing more than a weighting of plane waves.

It is clear that any aperturing of an optical beam will influence its angular spectrum. The most known effect of aperture limitation is the broadening of the spectrum due to the onset of higher spatial frequencies to describe hard edges, and a concomitant increase in oscillations of the light intensity after propagation. Similar arguments lead to the space bandwidth product, and the well known properties of Gaussian beam propagation, e.g., small beams, or aperture beams, have larger divergence and hence larger far field patterns, and vice versa. In the case of Bessel beams, its angular spectrum has the well-known ring-shaped profile, and as with the case of Gaussian beams, when the beam is apertured the ring width is broadened. A more nontrivial and interesting question arises: how does the angular spectrum change if the beam is bounded by a non-centered circular aperture? Since the angular spectrum arises due to a spatially localized part of the Bessel beam, we refer to this as the local spatial spectrum. This question has a straightforward relation to the so-called plane-wave representation of Bessel beams. According to this representation, the Bessel beam consists of a set of plane waves, wave vectors of which lie on the surface of a cone. This model was proposed by Durnin [1] to obtain a Bessel beam, and was implemented by placing a narrow circular diaphragm in the front focal plane of lens and illuminated by plane wave (Fig. 1).

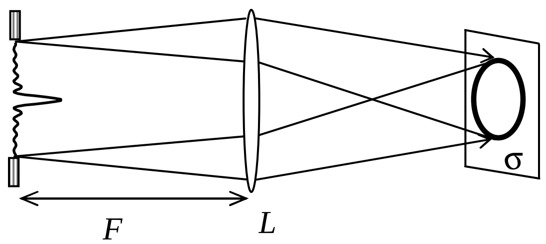


Figure 1. Optical Fourier transformation of Bessel beam.

In the back focal plane σ of lens L the annular field has a radius $R \approx \gamma F$ where γ is the cone angle of the Bessel beam. A superposition of plane waves in such a scheme is generated by imaginary point sources distributed within the circular diaphragm. When the diaphragm is placed outside the lens focus, the Bessel beam can be described as a superposition of lens-produced images of spherical waves formed by some effective circular source in the object plane of the lens [2]. In the axicon-based scheme of the Bessel beam production the appearance of a superposition of plane waves arises when analyzing the diffraction integral by the stationary phase method. At last, within the framework of rigorous solutions of Helmholtz equation, the Bessel beam can be presented as the superposition of two conical beams described by the zeroth- and first- order Hankel functions corresponding to convergent and divergent cylindrical waves [3]. It is essential to note that the above discussed models of Bessel beam lead to the ring-shaped angular spectrum. The aforesaid remarks concern the entire spectrum, i.e. the spectrum obtained for the beam as a whole. The entire spectrum appears experimentally in special non-localized physical processes such as the vector interaction in second harmonic generation and parametric frequency conversion [4, 5]. But the majority of linear and non-linear optical processes with Bessel beams are spatially localized, and consequently their behavior depends on the local spatial spectrum. Thus the challenge is to determine whether the ring-shaped angular spectrum persists when the limiting aperture is displaced from the center of the Bessel beam. This problem will be investigated for Bessel beams generated by conventional optical schemes, both experimentally and theoretically.

2. Fourier transformation of annular field

2.1. Theoretical results

An optical scheme, which is intended for studying the local spatial spectrum of Bessel beams, is shown in Fig. 2 [6]. An annular diaphragm D of radius R_{in} is used to generate a Bessel field in neighboring of back focal plane of lens L_1. The amplitude distribution of this field in the transverse plane is written as $a_1(\vec{r})$. A circular aperture with a transmission function $\sigma(\vec{r})$ is placed in the focal plane, at an arbitrary distance from the optical axis, and acts on this field. Then the output field $a(\vec{r})$ in may be written as

$$a(\vec{r};\sigma) = -\frac{i}{\lambda F_2}\int a_1(\vec{r}_1)\sigma(\vec{r}_1)\exp\left(-\frac{ik_0\vec{r}_1\vec{r}}{F_2}\right)d^2r_1 , \tag{1}$$

One can see that Eq. (1) has the form of a two-dimensional Fourier transform. The function $a(\vec{r};\sigma)$ will be referred to as the amplitude of the local spatial spectrum, with an intensity given by $I(\vec{r};\sigma) = |a(\vec{r};\sigma)|^2$.

It is clear from Fig. 2 that the amplitude $a_1(\vec{r}_1)$ is the Fourier transform of an input field

$$a_1(\vec{r}_1) = -\frac{i}{\lambda F_2}\int \tau(\vec{r}_2)\exp\left(-\frac{ik_0\vec{r}_1\vec{r}_2}{F_2}\right)d^2r_2 , \tag{2}$$

where $\tau(\vec{r}_2)$ is the transmission function of circular diaphragm *D*.

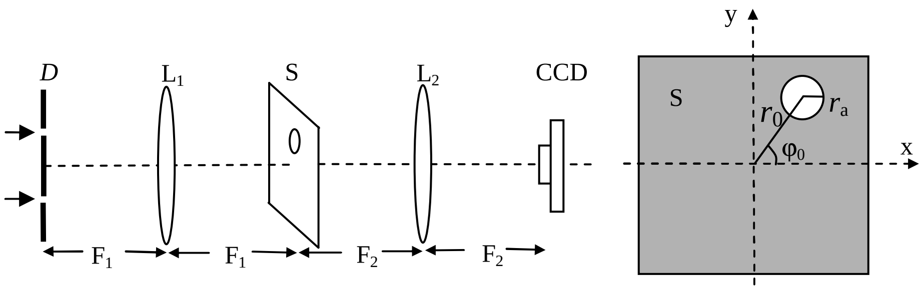


Figure 2. Optical scheme for measuring local spatial spectrum of Bessel beam generated when using annular diaphragm *D*. Here $L_{1,2}$ are Fourier transforming lenses, *S* is the screen with a circular hole of radius r_a shifted in the position $\vec{r}_0 = (r_0, \varphi_0)$.

It follows from Eqs. (1) and (2) that

$$a(\vec{r}) = -\frac{i}{\lambda F_1}\int \tau(\vec{r}_1)\tilde{\sigma}(\vec{r} + F_2\vec{r}_1/F_1)d^2r_1 , \tag{3}$$

where $\tilde{\sigma}(\vec{r})$ is the Fourier transform of the function $\sigma(\vec{r})$

$$\tilde{\sigma}(\vec{r}) = -\frac{ir_a}{r}J_1\left(\frac{k_0 r_a r}{F_2}\right)\exp\left(-\frac{ik_0\vec{r}_0\vec{r}}{F_2}\right). \tag{4}$$

On substituting Eq. (4) into Eq. (3), we obtain the following expression for output field:

$$a(\rho,\varphi) = \frac{-r_0}{\lambda F_1 F_2}\int_0^{2\pi}\int \tau(\rho_1)\frac{J_1(k_0 r_0 S(\rho,\rho_1,\varphi,\varphi_1))}{S(\rho,\rho_1,\varphi,\varphi_1)}\exp\left[\frac{ik_0\rho_0[\rho_1\cos(\varphi_1-\varphi_0)+\rho\cos(\varphi-\varphi_0)]}{-F_1}\right]\rho_1 d\rho_1 d\varphi_1 , \tag{5}$$

where $S(\rho,\rho_1,\varphi,\varphi_1) = \sqrt{\rho^2/F_2^2 + \rho_1^2/F_1^2 + 2\rho_1\rho/F_1 F_2\cos(\varphi-\varphi_1)}$, $J_1(x)$ is the first order Bessel function.

This equation allows one to calculate the local spatial spectrum of the aperture Bessel beam (in this case generated using an annular diaphragm). If the diaphragm has a narrow enough width, the transmission function $\tau_0(\rho)$ can be approximated as a delta function: $\tau_0(\rho) = R_{in}\delta(\rho - R_{in})$, where R_{in} is the diaphragm radius. This allows Eq, (5) to be simplified to yield:

$$a(\rho,\varphi) = -\frac{r_0 R_{in}^2}{\lambda F_1 F_2}\int_0^{2\pi}\frac{J_1(k_0 r_0 S(\rho,R_{in},\varphi,\varphi_1))}{S(\rho,R_{in},\varphi,\varphi_1)}\exp\left[\frac{ik_0\rho_0[R_{in}\cos(\varphi_1-\varphi_0)+\rho\cos(\varphi-\varphi_0)]}{-F_1}\right]d\varphi_1 . \tag{6}$$

2.2 Experiment and numerical calculations

A ring diaphragm of diameter D = 12 mm and ring width 80 μm was produced by photolithography. The diaphragm was placed in the front focal plane of the lens L_1 of focal length F_1 = 50 cm. The perturbing diaphragm (S) consisted of a circular aperture of radius r_0 = 185 μm (Fig. 2) placed at normal incidence to the field. A second inverse Fourier transforming Lens L_2 of focal length F_2 = 20 cm was used to produce the desired local spatial spectrum, the intensity of which was measured on a standard CCD camera.

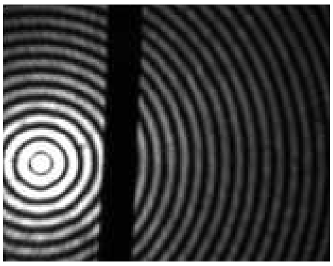


Figure 3. Fragment of Bessel beam produced in back focal plane of lens L_1. For estimating the typical scale of Bessel beam , the dark strip width is shown to be 80 µm.

Fig. 4 shows the results of measuring the spatial spectrum obtained at various positions of the circular diaphragm relative to the optical axis.

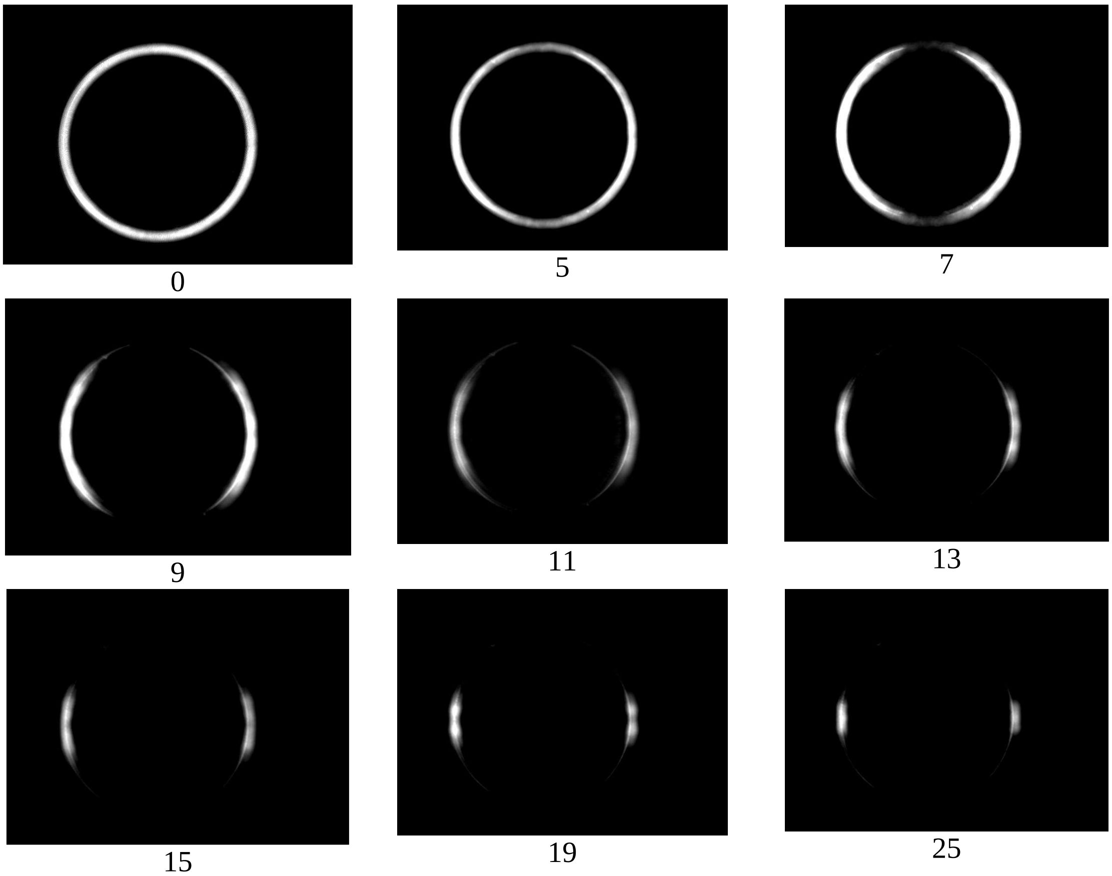
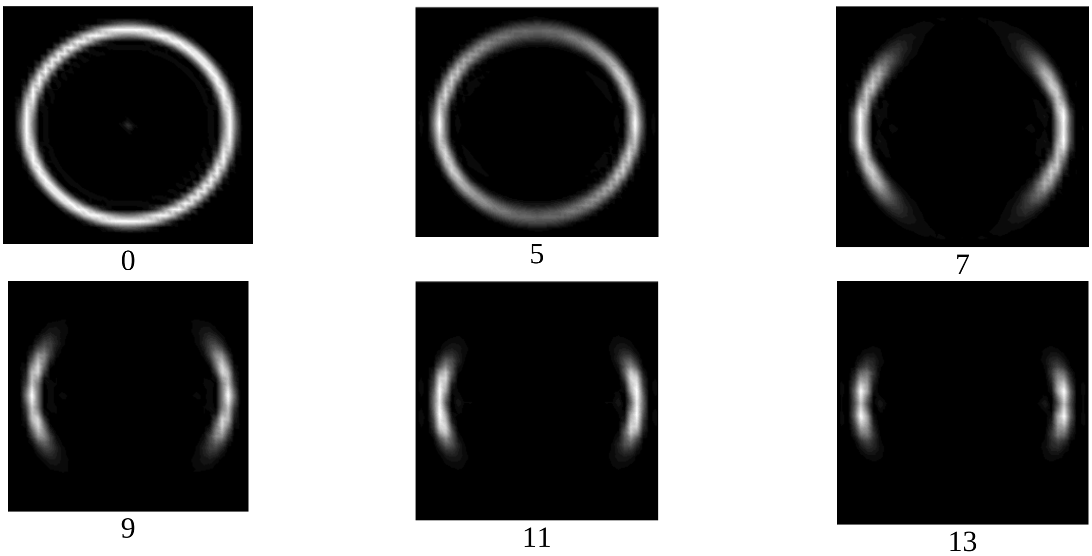


Figure 4. Characteristic changes of the local spatial spectrum of a Bessel beam caused by the off-axis shift of the circular diaphragm. The shift value is equal to $r_0 = (30\,m)$ µm where *m* is the number under the figure.

Figure 4 shows the change in the local spatial spectrum of the Bessel beam due to an off-axis shift of the circular opening. Namely, the initial ring- full spectrum is transformed into a pair of arcs whose angular size reduces gradually. The corresponding theoretical predictions for the same conditions (using Eq. (6)) are shown in Fig. 5. It is clear that there is very good qualitative comparison.



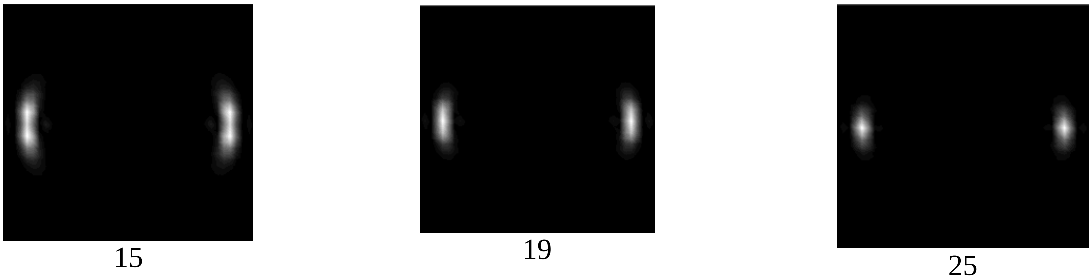


Figure 5. Theoretically investigated intensity distribution of the local spatial spectrum of the Bessel beam. The shift value is the same as in Fig. 4.

These results, as presented in Figs. 4 and 5, show that, as a rule, the local spatial spectrum of a Bessel beam is not an annular ring. The exception takes place when the center of the circular aperture is situated on the axis of the beam, or when the shift is so small that central lobe of the beam lies inside the opening. Consequently, annular form of the angular spectrum is realized only in a relatively narrow near-axial region of the beam. Outside this region, the angular spectrum of the perturbed beam takes the shape of a pair of arcs whose azimuthal orientation is determined by the orientation of the diaphragm relative to the beam axis. Superpositions of such azimuthally orientated local spatial spectra return the well-known annular pattern.

These results allow one to correct the so-called plane wave interpretation of Bessel beams. The matter concerns the representation of Bessel beams as composed of an array of plane waves whose wave vectors lay on the cone surface (see, for example, [1]). It is obvious that such an interpretation is not correct as applied to an arbitrary beam area except for its central region. In off-axis regions of the Bessel beam the wave vectors of plane waves fill only a part of the circular cone. At a sufficiently large shift from the beam axis the spatial spectrum profile approaches a double-spike one (see last frames of Figs. 4, 5), which corresponds to the approximation of two plane waves.

More detailed information about such fields can be obtained from appropriate 1D- sections of two- dimensional intensity profiles. Fig. 6 shows how the intensity of an arc-type field varies with the azimuth angle φ. Note that these graphs correspond to the maximal value of intensity in radial direction.

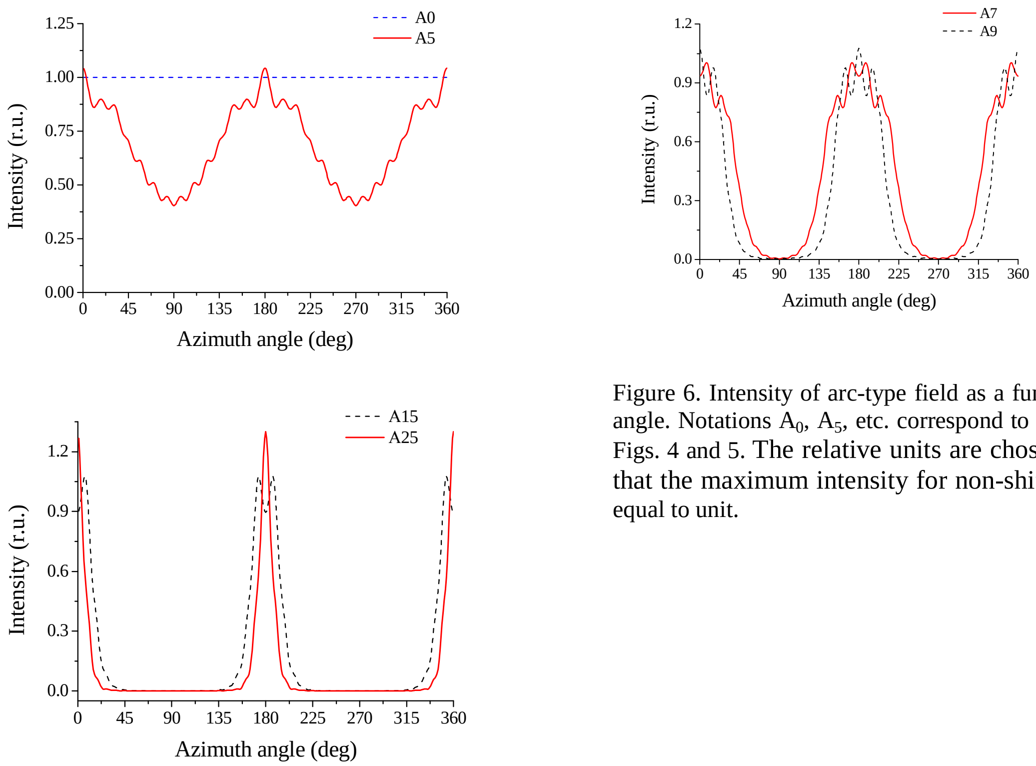


Figure 6. Intensity of arc-type field as a function of azimuthal angle. Notations A_0, A_5, etc. correspond to images 0, 5, etc. in Figs. 4 and 5. The relative units are chosen in such a way that the maximum intensity for non-shifted diaphragm is equal to unit.

The uniform intensity distribution of the field corresponds to the non-shifted diaphragm. At a nonzero shift of the circular window there appears a large-scale double-peak modulation. The width of these peaks decreases with increasing the shift. It is interesting to point out that the origin of the small-scale modulation on the graphs is not yet clear.

3. Local spatial spectrum of the Bessel function

It is interesting to investigate the local spatial spectra of the Bessel function irrespective of how it is produced optically. Consider then the zero-order Bessel function $J_0(q\rho)$. When the diaphragm shift $r_0 < r_a$, (see Fig. 2), the spectral amplitude $a(\rho,\varphi)$ can be calculated from

$$a(\rho,\varphi)=2\pi\int_{0}^{r_a-r_0}J_0(k_0\gamma\rho_1)J_0\left(\frac{k_0\rho\rho_1}{F}\right)\rho_1 d\rho_1+\int_{r_a-r_0}^{r_a+r_0}J_0(k_0\gamma\rho_1)D(\rho,\rho_1,\varphi)\rho_1 d\rho_1\,, \tag{9}$$

where

$$D(\rho,\rho_1,\varphi)=\int_{\varphi_0-\Delta\varphi(\rho_1)}^{\varphi_0+\Delta\varphi(\rho_1)}\exp\left(-\frac{ik_0\rho\rho_1}{F}\cos(\varphi-\varphi_1)\right)d\varphi_1\,, \tag{8}$$

and $\Delta\varphi(\rho)=\arccos[(\rho^2+r_0^2-r_a^2)/2r_0\rho]$.

Fig. 7 shows the intensity of the local spatial spectrum as a function of the azimuth angle at different shifts of the diaphragm.

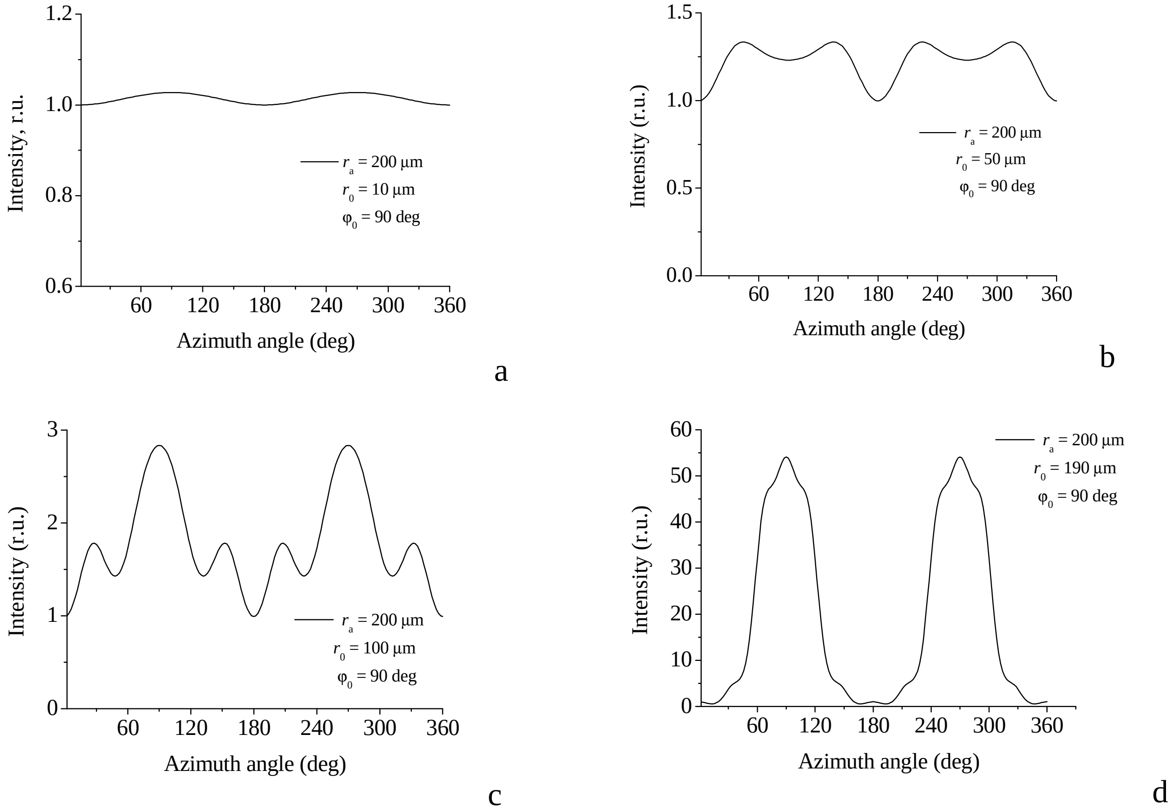


Figure 7. Local spatial spectrum of Bessel function as a function of the azimuth angle. The relative units are chosen in such a way that the intensity at start point φ = 0 deg is equal to unit. Cone angle γ = 0.25 deg; r_a = 200 μm; F = 0.5m; φ_0 = 90 deg; r_0 = 10 μm (a), 50 μm (b), 100 μm (c), 190 μm (d).

When the circular diaphragm is situated outside the beam axis, the spectrum is calculated to be

$$a(\rho,\varphi)=\int_{r_0-r_a}^{r_0+r_a}J_0(k_0\gamma\rho_1)D(\rho,\rho_1,\varphi)\rho_1 d\rho_1\,, \tag{7}$$

Fig. 8 illustrates the property of the local spectrum in this case.

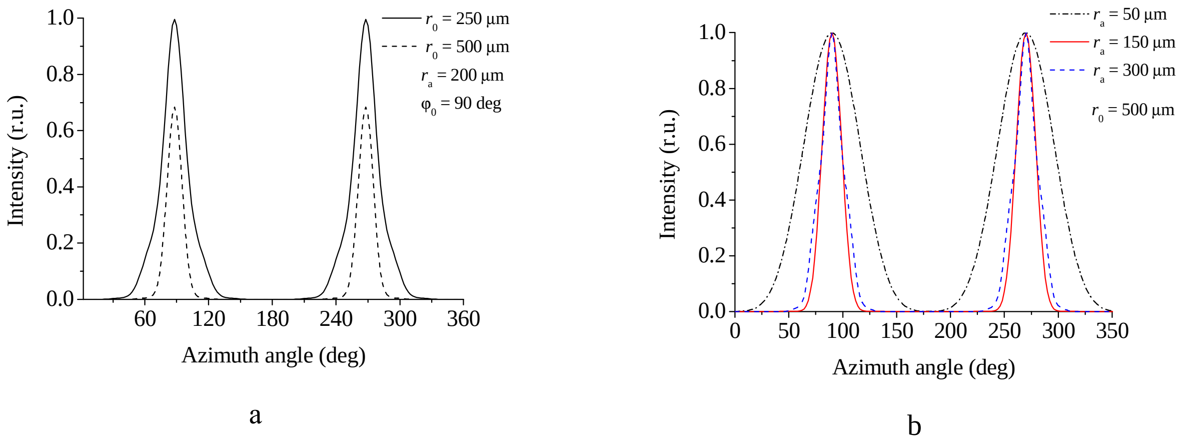
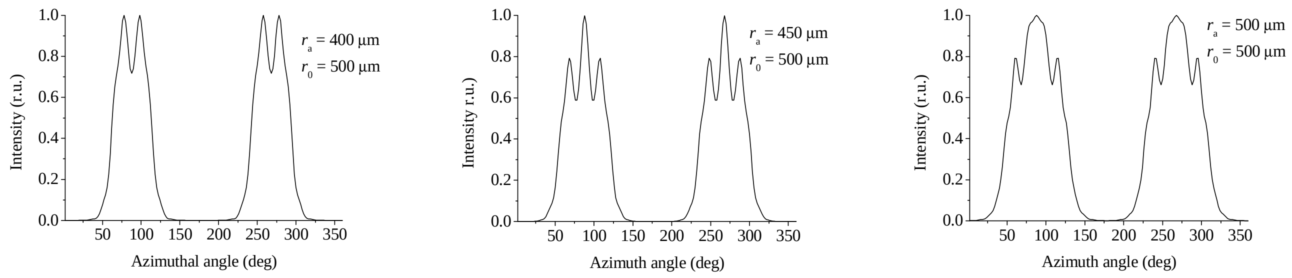


Figure 8. Dependence of the intensity of the local spectrum of the Bessel function on the azimuthal angle at different values of shift r_0 of annular aperture (a) and from its radius R (b). The relative units are assumed were the maximal intensity is equal to unite. Cone angle is γ = 0.25 deg., F = 0.5m.

It is seen that the local spatial spectra of the Bessel function has the arc-type structure similar to the obtained in sect. 2.2. The maximal intensity occurs at $\varphi = \varphi_0$ and $\varphi = \varphi_0 + \pi$ and the spectrum width decreases with increasing the aperture shift r_0 (Fig. 8a). The dependence of the spectrum width on the diaphragm radius r_a is more complex (Fig. 8b). At small r_a the spectrum is wide and narrows with increasing r_a. This behavior is explained by the influence of diffraction at the edge of the circular aperture. As r_a is increased further, so the spectrum is broadened and begins to deform. Such behavior takes place when the diaphragm edge approaches the centre of the beam (Fig. 9). In the region of maximum intensity an oscillating behavior appears with two and more local maxima.



a b c

Figure 9. Spectrum transformation when the edge of the circular aperture is approached to the beam center.

From this we conclude that the local spatial spectrum of the Bessel function is not annular. The presence of two global maxima means that the light energy inside a Bessel beam propagates preferably in two directions. The azimuth angle of these directions, as it was pointed out above, is equal to $\varphi = \varphi_0$ and $\varphi = \varphi_0 + \pi$. When the value of r_0 is increased, the two waves approach plane waves as their local spatial spectrum is converged into two points. This can be explained by considering the circular aperture as passing two components of the underlying conical waves. One of these waves is convergent and the other divergent [3].

4. Scheme based on axicon

Analogical investigations of the local spatial spectra were conducted for Bessel beams produced by an axicon. A refractive axicon with a base angle of about 1.7 deg was illuminated by a Gaussian beam of waist radius w_0 = 6 mm. The circular opening S (Fig. 2) with a radius of 185 μm was placed after the axicon at the distance of maximal diameter of the Bessel beam. The local spatial spectrum was generated using a Fourier transform lens with a focal length of F = 0.3 m.
Here the transformation of the annular spectrum into an arc-type one was also observed when increasing the shift of the circular aperture (see Figs. 10, 11 and 12). The difference here is that the appearance of asymmetry of the arc-type spectrum, which increases when the shift becomes larger.

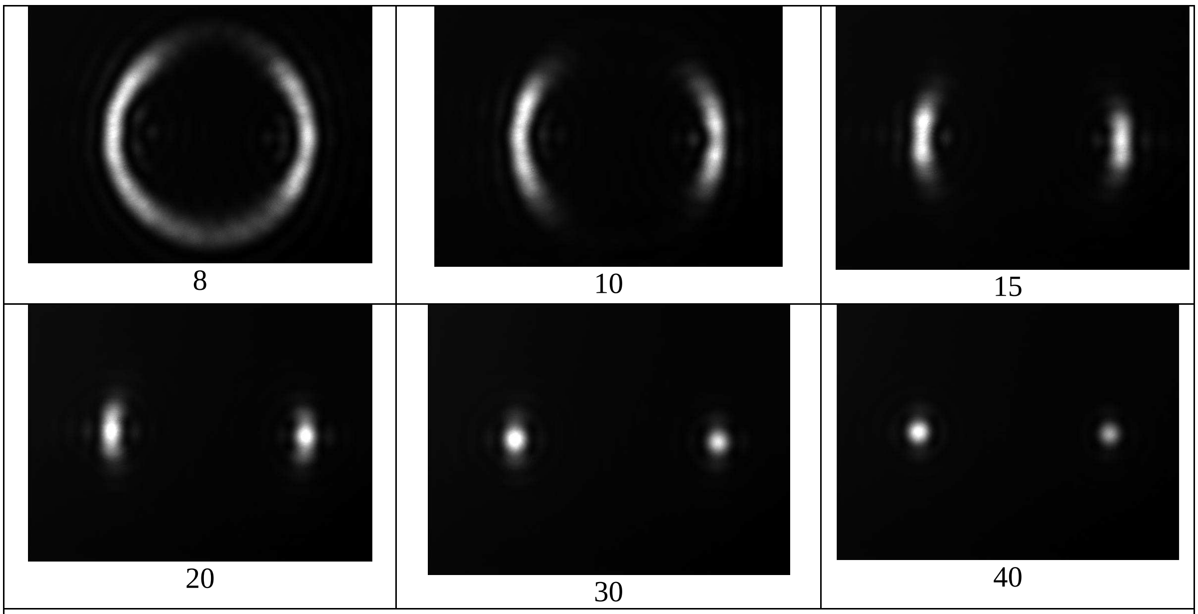


Figure 10. Local spatial spectra of Bessel beam produced by the axicon. The transverse shift of the circular diaphragm equals the number under the figure multiplied by 30 μm.

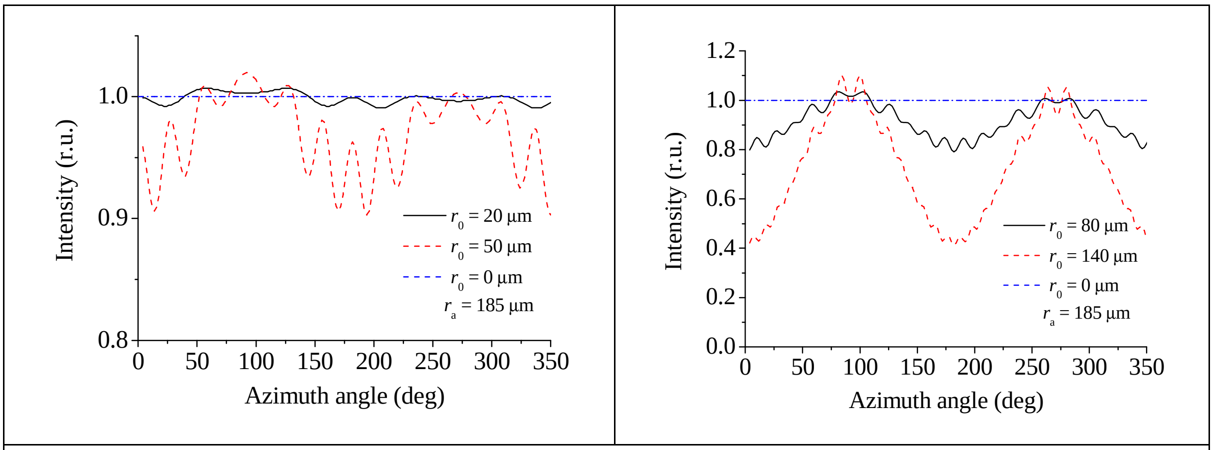


Figure 11. The calculated intensity dependences on the azimuth angle of the local spectrum of an Bessel beam formed by the axicon. The case is illustrated where the beam axis is within the circular aperture. The parameters used: cone angle γ = 0.9 deg., focal distance of the lens *F* = 0.3m, aperture radius R = 185 mkm, half-width *w* = 6 mm.

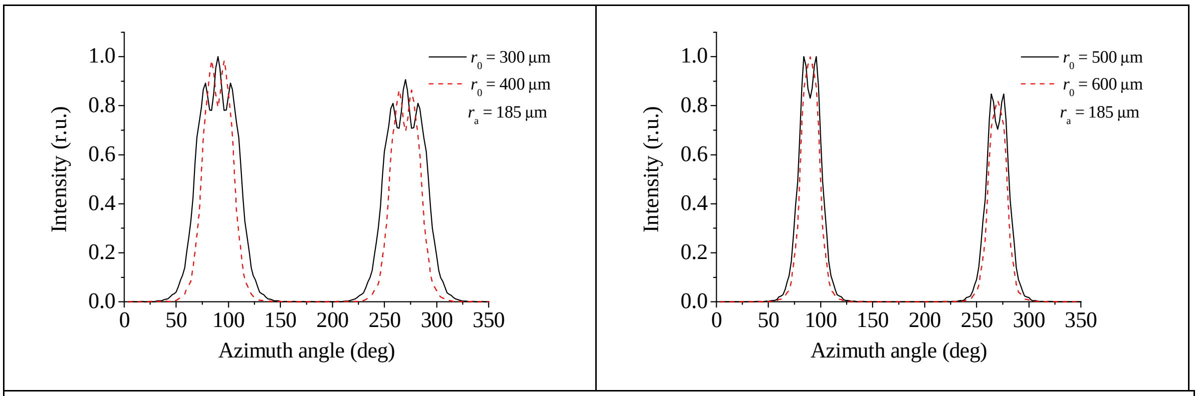


Figure 12. The calculated azimuthal intensity dependences of local spatial spectrum of Bessel beam produced by axicon. The case where the position of the circular aperture is outside the beam axis. The parameters used: cone angle γ = 0.9 deg., focal distance of the lens *F* = 0.3 m, aperture radius R = 185 mkm, w_0 = 6 mm.

The asymmetry of the spectrum can probably be explained by the fact that the beam produced by an axicon differs slightly from a Bessel function, and rather approximates a Bessel-Gaussian beam. Mathematically this difference originates from an unequal contribution of two stationary points into the diffraction integral. That leads to some differences in amplitude of the convergent and divergent conical waves that compose the Bessel-Gaussian beam.

It is easy to obtain, by calculating the Freshnel diffraction integral using the stationary- phase method, that the field after the axicon is given by:

$$a(\rho,z)=\left[f_+(\rho,z)J_0(k_0\gamma\rho)-if_-(\rho,z)J_1(k_0\gamma\rho)\right], \tag{10}$$

where $f_\pm(\rho,z)=\left(f_1(\rho,z)\pm f_2(\rho,z)\right)/2$, $f_{1,2}(\rho,z)=\sqrt{1\pm\frac{\rho}{\gamma z}}\exp\left(-\frac{(\gamma z\pm\rho)^2}{w_0^2}\right)$.

To numerically calculate the local spectrum, Eqs. (7) and (9) were used with the replacement of $J_0(k_0\gamma\rho)$ into $a(\rho,z)$ from Eq. (10). The results are shown in Figs. 11 and 12, where the direction of the shift of the circular aperture is φ_0 = 90 deg. As is seen, for a near-axial position of the aperture (Fig. 11), when increasing of r_0 there is a gradual transformation of the homogenous annular field into two arcs, as described earlier. At an off-axial position of the aperture, the azimuthal inhomogeneity of the spectrum is evident. Here the asymmetry of the intensity of two arcs is also observed. This asymmetry is seen to be more essential at the periphery of the Bessel beam (Fig. 12) than in its central area (Fig. 11). It corresponds exactly to the studied earlier field structure behind the axicon, namely, the contribution of the component containing the Bessel beam of the first order in Eq. 10 increases when the aperture moves off the beam axis.

5 Conclusions

We have shown that the angular spectrum of a Bessel beam apodized by a circular aperture depend considerably on the position of the aperture relative to the optical axis. If the shift is large enough then the central lobe of a beam lies outside the opening, and the shape of local spatial spectrum becomes arc-like, i.e., composed of two bright points rather than an annular ring. The width of the arcs in the azimuthal direction decreases with increasing shift, and the orientation of the arcs rotates when rotating the aperture about the optical axis. This property of the local spatial spectrum is the intrinsic feature of Bessel functions irrespective of the method of the Bessel beam generation.

The arc-like structure of the local spatial spectrum is an important property of Bessel beams, and is necessary to be taken into account when interpreting experimental results obtained with Bessel beams. The simple example is the application of Bessel beams for profilometry of cylindrical surfaces [7, 8]. The shift of the tested cylindrical object with respect to the optical axis can lead to an azimuthal modulation of the optical field illuminating the cylinder surface. At rather large shifts a part of the cylindrical surface is not illuminated at all. The second example concerns the formation of a one-dimensional array of particles in Bessel beam based optical tweezers [9]. The efficiency of manipulation of microparticles here can essentially depend on the self-reconstruction of a field, screened by these particles [9]. From the local spatial spectrum of Bessel beams it follows that the conditions of recovery of the Bessel beam behind a particle will be changed when this particle has a non-zero transverse shift relative to the beam axis.

References

[1] J. Durnin, J.Opt., “Exact solution for nondiffracting beams. I. The scalar theory”, Soc.Am. **B 4,** 651 (1987).

[2] Z.Bouchal, J.Wagner, M. Olivík, “Bessel beams in the focal region”, Opt. Eng. **34,** 1680-1688 (1995).

[3] S. Chavez-Cerda, G.S. McDonald, G.H.C. New, “Nondiffracting beams: travelling, standing, rotating and spiral waves”, Opt. Communs. **123**, 225-233 (1996).

[4] V. N. Belyi1, N. S. Kazak, N. A. Khilo, “Frequency conversion of Bessel light beams in nonlinear crystals”, Quantum Electron **30,** 753-766 (2000).

[5] R. Gadonas, V. Jarutis, A. Marcinkevicius, V. Smilgevicius, A. Stabinis, “Angular distribution of second harmonic radiation generated by Bessel beam”, Opt. Communs. **167**, 299-309 (1999).

[6] V.Belyi, N.Khilo, A.Forbes, A.Ryzhevich, Generation and propagation of high-order Bessel vortices in linear and non-linear crystals. Proc. SPIE, V.7430, p. 7430O0F-1-8 (2009)

[7] T. Dresel, J. Schwider, A. Wehrhahn, S. Babin, “Grazing-incidence interferometry applied to the measurement of cylindrical surfaces”, Opt. Eng. **34**, 3531 – 3535 (1995).

[8] N. Khilo, V. Belyi, N. Kazak, A. Mashchenko, P. Ropot, “Conical beam-based laser profilometer for testing roller bearings”, Proc. SPIE **6616**, 66162L -01 – 66162L-08. (2007)

[9] D. McGloin and K. Dholakia,”Bessel beams: diffraction in a new light”, Contemp. Phys. **46**, 15–28 (2005).