Simple Preconditioning Technique: Empirical Formula for Condition Number Reduction at a Junction of Several Wires

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Abstract

The condition number for a method-of-moments' impedance matrix resulting from a junction of several wires is frequency dependant and can be minimized at a given frequency using several approaches. An empirical formula for an optimum, condition-number minimizing selection of the dependent variable for a multiplet on the junction is presented. The results indicate possibility to lower the condition number by an order of magnitude with virtually no computational overhead. The results may also be used to mesh the structure so as to minimize the condition number.

1. Introduction

Many computational methods, including the method of moments [1-3] used in this work, result in a system of linear equations. The feasibility to solve such system, as well as the accuracy of the solution, depends on the condition number of the system [4-8]. The condition number reflects the numerical stability of the system: if the condition number is large, the system is ill-conditioned, whilst a value near unity indicates feasibility of a reliable solution. For a direct solution, the accuracy of the solution is inversely proportional to the condition number of the impedance matrix [5, 9].

There exist several methods to precondition a system of linear equations and reduce the condition number, see for example [5, 8, 10, and 11]. These methods are computationally intensive, as they normally require matrix operations. On the other hand, it was shown [4, 6], that it is possible to reduce the condition number or limit it, at least for the thin wire electrical field integral equation (EFIF) formulation [3, 12]. The principle is based on the possibility to choose the dependent variable from the Kirchhoff current law (KCL), applied to a junction of several wire segments. The choice is arbitrary due to blindness of the indexes used to number the wires, as illustrated with Figure 1. Having three wire segments, one can number them in six possible ways. This also corresponds to six possible values of the condition number [6], although not all values are unique.

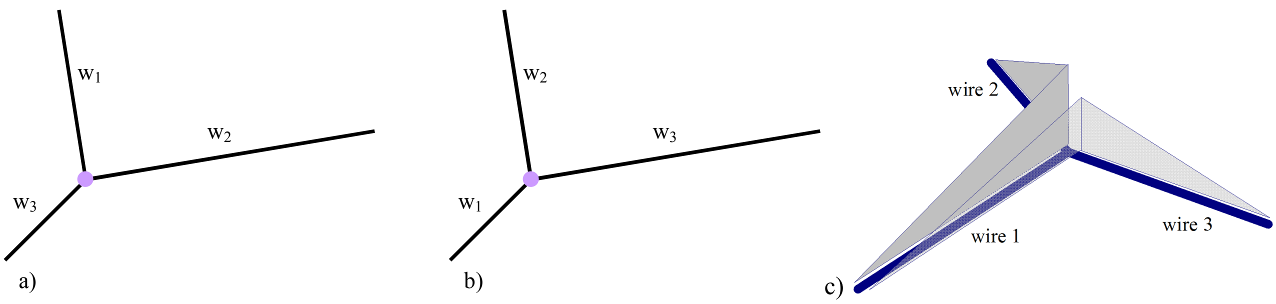

Figure 1. Numbering of wires attached to a junction. Two out of six combinations are shown in (a) and (b). Each wire is denoted with the symbol *w* and a unique number. Picture in c) shows one possible assignment of basis functions where w_1 is the reference wire.

In the method limiting the condition number from reaching the maximum value [4], the condition number can be prevented from being unnecessarily large without any computations, by an appropriate choice of the Kirchhoff current law (KCL) – based multiplet decomposition [3] performed on a junction of several wire segments. In the same publications [4, 6], it was also shown that, at an expense of iterating through solving for all possible combinations in the selection of the current decomposition, it is possible to reduce the condition number to its lowest value for a given geometry. Both methods work by identifying the wire segment which is associated with the dependant unknown in the multiplet [3] current decomposition performed on a junction with several wires attached. It was also shown that reducing the condition number for individual parts of a complex system helps to reduce the condition number for the whole system.

This paper considers two trial scenarios and by means of numerical experiments proposes a method to select the dependent unknown in such a way as to obtain the condition number close to its minimally possible value. An empirical formula is proposed to calculate the length of the wire segment to give this nearly optimum result. No previous papers showing such a formula has been found.

The paper is organized in the following manner. Section 2 introduces the components of the method proposed. Section 3 discusses the numerical experiments performed and shows the derivation of the empirical formula. Finally, the Section 4 concludes the paper.

2. Method of Investigation

A structure composed of thin wires was especially devised to highlight the dependence of the condition number on the choice of the dependant variable in a multiplet. This involved multiple wire segments attached and electrically connected to the same point (junction).

The first structure used for running the modeling is shown in Figure 2 and defined in Table 1. The figure shows six wire segments connected to a junction at the node n1. The wire segments have different lengths, enabling 12:1 range (in terms of the ratio of the maximum length to the minimum length) in the length of the segments with six levels of length.

The second model had a similar geometry but a larger, 100:1 range of the lengths of the wire segments.

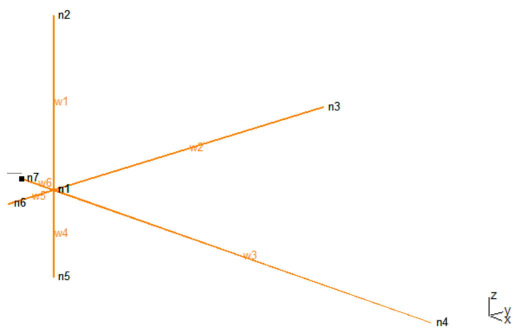

Figure 2. Geometry of the model (shown as a print-screen from WIPL-D [12]).

Table 1. Definitions for the wire structure (radii of all wires was 1 mm)

Wire no.	Beginning (x,y,z) [m]	End (x,y,z) [m]	Length of wire [m]	Comment
1	(0,0,0)	(0,0,1)	1	
2	(0,0,0)	(0,2,0)	2	
3	(0,0,0)	(3,0,0)	3	Longest
4	(0,0,0)	(0,0,-1/2)	1/2	
5	(0,0,0)	(0,-1/3,0)	1/3	
6	(0,0,0)	(-1/4,0,0)	1/4	Shortest

The method used to study the dependence of the condition number on the choice of dependant unknown (i.e. on the choice of doublet and associated wire segment; hereinafter, this wire segment is referred to as the reference wire) includes the following steps:

Loop 1: through all the frequencies in a pre-defined range
    Loop 2: through all the wire segments attached to the junction
        Select the dependent variable on the current wire segment
        Compute impedance matrix and condition matrix
    End of loop 2
End of loop 1

The dependent variable is associated with a particular wire segment. In each iteration of the loop 2, a different wire segment was chosen as the *reference wire*, i.e. the wire on which the dependent variable is defined. Please see Figure 1c for an example of assignment.

After the computations are done, the computed condition numbers are inter-compared, enabling to analyze the length of the wire segment associated. This process is illustrated for example in [4, 6]. The final analysis was performed in terms of wavelength, leading to an empirical formula.

The post-processed results, such as input admittance on several pre-defined ports and current distribution were calculated by the code [7] and validated by comparing them to the results produced by a commercial program WIPL-D [12].

3. Results and Discussions

The key result of the simulations on the first model is shown in Figure 3. Figure 3a shows the dependence of the value of the condition number of the impedance matrix against frequency. Each curve corresponds to a particular choice of the reference wire. The plots show that the condition number is frequency dependent. The *positions* of the peaks and the troughs among the curves are the same with respect to frequency, and attributed to the resonances seen by the structure. The critical part in the plot lies in the comparison between the *values* of the condition number among the different curves at any fixed frequency. It is clear that using the shortest wire (w_6) as the reference wire at low frequencies leads to the highest condition number. In order to minimize the condition number at those frequencies, the longest wire (here: w_3) should be used instead. The opposite situation holds at the high frequencies: in order to minimize the condition number, one needs to take the shortest wire (here: w_6) as the reference wire.

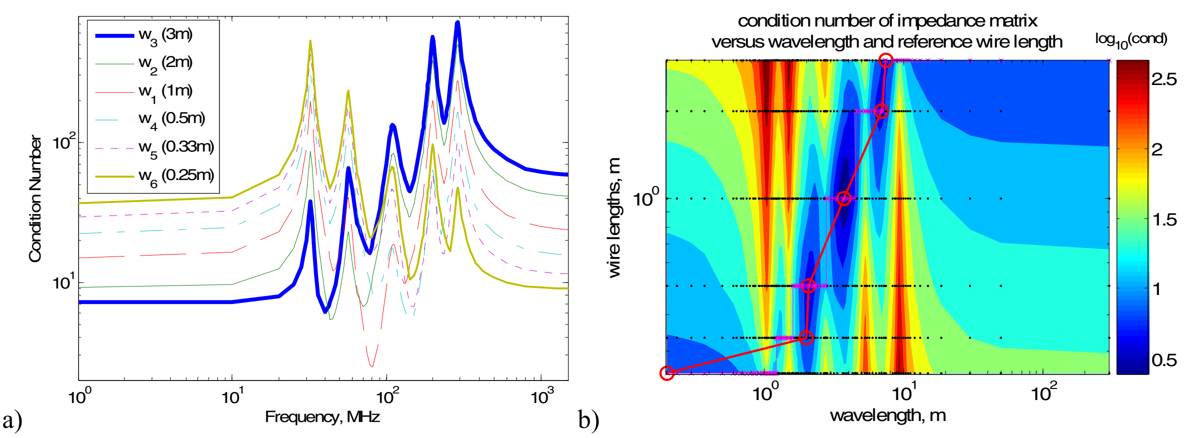

Figure 3. a) Condition number versus frequency: various choices of reference wire (shown as w_n in the legend); b) Interpolated distribution of the condition number against both wavelength and the length of the reference wire. Small dots indicate the discrete positions at which a numerical model was run. Circles ○ joined with a red line show at which wavelength the condition number is minimal for each wire. Crosses × joined with a magenta line show the selection of the reference wire that minimises the condition number, at each wavelength (frequency).

As an observation, one may notice that the number of resonances supported by the numerical model is limited to six, contrary to the infinite number of resonances to be observed in a real physical structure. This is due to the type and quantity of the basis functions used in the numerical modeling. The *doublet* [3, 12] (commonly referred to as *roof-top* [2]) basis functions were used. This corresponds to using *one* triangularly profiled basis function per each wire segment. Such set of basis functions (referred in [3] to as a *multiplet*) can model the current distribution at low frequencies but is incapable of supporting higher frequencies. Nevertheless, as the doublets form a part of the hierarchical higher-order basis functions [3, 12], they contribute to the value of the condition number for the whole system. Thus, it is important to minimize the condition number for the multiplet.

At the frequencies in the middle of the resonance region shown in Figure 3a, the wire segments w_1 and w_3 (with length intermediate among all the wire segments) are those giving the lowest condition number. Moreover, the minimum value of the condition number for each resonance shown in Figure 3a corresponds to a different wire segment, and this goes in the order of the wire segment's length. It is also possible to observe that the ratio of the maximum condition number to minimum condition number at any given frequency may reach ten, corresponding to saving a whole digit of accuracy.

The data from Figure 3a was plotted in Figure 3b in a parametric presentation and against wavelength λ instead of frequency [6]. This permitted to make a straight line fit through the resonance region and thus to come up with the following empirical formula enabling to estimate the length of the reference wire L_{opt} minimizing the condition number:

$L_{opt}(\lambda) = 0.45\cdot\lambda - 0.41$. The same procedure was repeated on the second model (with a larger difference in the lengths of the wire segments). More data points were used, and this resulted in a similar yet possibly more accurate expression:

$$L_{opt}(\lambda) = 0.5\cdot\lambda - 0.47 \quad (2)$$

An estimation of the lowest and highest resonance frequencies for a structure where several wire segments are joined at one point can be made [6]. This permits one to use the shortest/longest wire segments at the frequencies respectively higher/lower than those estimated lowest and highest resonance frequencies. This may be considered as the asymptotic cases. In between the frequency bounds from [6], one can use the expression (2) to calculate the length of the wire which will lead to the lowest possible condition number. This approach is nearly as simple as the one described in [4, 6] and permits not just to limit the condition number but to lower it further.

4. Conclusion

A numerical study was performed on a structure comprised of several thin wire segments; a structure which is the basis to a multiplet type of current decomposition in the moment method. The study has shown that the condition number of the moment method impedance matrix due to such a structure is frequency dependent. It has also been shown that the dependence may lead to an empirical formula for calculating the desired length of the reference wire. Using this formula and limit/asymptotic cases discussed in the paper, the condition number due to the multiplet can be minimized at any given frequency. In addition, it is expected that the formula may be used in meshing a structure to obtain the mesh that minimizes the condition number. It is also expected that the method proposed will also apply to triangular, quadrilateral and other surface elements.

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