Evolution of optical vortex distributions in stochastic vortex fields

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ABSTRACT

Stochastic vortex fields are found in laser speckle, in scintillated beams propagating through a turbulent atmosphere, in images of holograms produced by Iterative Fourier Transform methods and in the beams produced by certain diffractive optical elements, to name but a few. Apart from the vortex fields found in laser speckle, the properties and dynamics of stochastic vortex fields are largely unexplored. Stochastic vortex fields with non-equilibrium initial conditions exhibit a surprisingly rich phenomenology in their subsequent evolution during free-space propagation. Currently there does not exist a general theory that can predict this behavior and only limited progress has thus far been made in its understanding. Curves of the evolution of optical vortex distributions during free-space propagation that are obtained from numerical simulations, will be presented. A variety of different stochastic vortex fields are used as input to these simulations, including vortex fields that are homogeneous in their vortex distributions, as well as inhomogeneous vortex fields where, for example, the topological charge densities vary sinusoidally along one or two dimensions. Some aspects of the dynamics of stochastic vortex fields have been uncovered with the aid of these numerical simulations. For example, the numerical results demonstrate that stochastic vortex fields contain both diffusion and drift motions that are driven by local and global variations in amplitude and phase. The mechanisms for these will be explained. The results also provide evidence that global variations in amplitude and phase are caused by variations in the vortex distributions, giving rise to feedback mechanisms and nonlinear behavior.

Keywords: Infinitesimal propagation equation, entangle photons, atmospheric turbulence, orbital angular momentum, decoherence

1. INTRODUCTION

The properties of optical vortices[1] (i.e. phase singularities found in optical fields) have been studies extensively.[2] As topological defects they carry integer topological charge and obey topological conservation laws. The topological charge can to some extent be associated with the orbital angular momentum in the beam.[3–5] Various aspects of optical vortices, such as their trajectories[6–19] and morphology[16, 20–22] have also been studied. However, these studies largely dealt with a small number (one or two) vortices in deterministic beams.

There are fewer studies of large collections of vortices. Some work on vortex arrays do exist.[23–26] In these studies the high degree of symmetry allows one to obtain full analytical solutions. When such a symmetry does not exist the problem becomes akin to a many-body problem. For such problems it is more useful to employ statistical methods. In such cases one would consider vortex fields rather than individual vortices.

Previous statistical investigations of vortex fields are restricted to the investigation of vortices in fully developed speckle fields, also called random optical fields.[27–32] These include investigations of the distribution of the optical vortex parameters[29] and the topology of the vortex trajectories.[33] Statistical optics[34] has been used as a fruitful approach for analytic investigations into the properties of random vortex fields.[27, 31]

Here we want to propose a new field of investigation, which we'll refer to as *stochastic singular optics*. This does not only include the statistical work on the vortex distributions in random optical fields, but also artificially generated vortex distributions. Such quasi-random vortex distributions can be produced with phase-only diffractive optical elements or by arranging speckle beams to be combined such that they interfere with each other in particular ways. One can also consider scintillated beams propagating through a turbulent atmosphere as an example of such stochastic vortex fields because they are not in equilibrium and are often not homogeneous.

There is a large variety of different initial conditions that one can impose on the vortex distributions. Usually the initial conditions represent an instability that decays to zero during propagation. The trancient behaviour that one observes during this decay process reveals the dynamics of the vortex distributions. The various initial conditions lead to a rich variety of such observations. Here we present a collection of these observations and present some initial attempts to explain them. In Section 2 a theoretical framework for this work is provided. The scenarios that are considered fall into three categories, which are presented in turn: homogeneous vortex fields in Section 3, inhomogeneous vortex fields with one-dimensional variations in Section 4, and inhomogeneous vortex fields with two-dimensional variations in Section 5. A summary is provided in Section 6.

2. FRAMEWORK

One can represent the vortices in stochastic vortex fields either in terms of the number densities, $n_p(x, y, z)$ and $n_n(x, y, z)$, of positive and negative vortices, respectively, or in terms of the combined vortex density $V(x, y, z)$ and the topological charge density $D(x, y, z)$. The two different ways to represent the vortices are related by

$$V(x, y, z) = n_p(x, y, z) + n_n(x, y, z) \quad (1)$$

$$D(x, y, z) = n_p(x, y, z) - n_n(x, y, z). \quad (2)$$

The number densities give the local expectation value for the quantity (such as the number of vortices) per unit area on any plane perpendicular to the direction of propagation (z-coordinate). On a given plane the number density varies as a function of the transverse coordinates x and y, and this function also varies from plane to plane as a function of z. The positive, negative and combined vortex densities are non-negative functions. On the other hand, the topological charge density can be positive or negative. It is important to note that the optical fields contain more information than is represented in terms of these vortex densities. Therefore, one can expect that the evolution of the vortex densities would be governed by more than just the interactions of the vortex densities on themselves and each other.

There is a limitation on how large the local topological charge density in a particular region can be. The net topological charge T inside a convex area with circumference C must be less than the number of times that the wavelength λ fits into the circumference $T < C/\lambda$, otherwise the bulk of the light on the circumference would be evanescent and not propagating. This limitation imposes a strong condition on homogeneous vortex fields. In the homogenous case the number densities are all independent of the transverse coodinates. The limitation implies that the positive and negative vortex densities must be equal [$n_p(z) = n_n(z)$] and by implication that the topological charge density in the homogeneous case is always zero [$D(z) = 0$].

The random vortex field found in a speckle field represents a special homogeneous case, because it remains unchanged during propagation — independent of z. One can say that such a field represents a state of equilibrium. Most non-equilibrium cases eventually evolve toward this equilibrium state. The properties of the equilibrium state therefore play an important role in all stochastic vortex fields.

One can divide the different cases of non-equilibrium initial conditions into different groups. First one distinguishes between homogeneous and inhomogenous initial conditions. For the inhomogeneous cases one can further divide them into one-dimensional variations and two-dimensional variations. Each of these are now discussed in turn.

3. HOMOGENEOUS FIELDS

A speckle field (which represents a homogenous vortex field in equilibrium) can be expressed as the product of a real-valued amplitude function, a continuous phase factor and a singular phase factor, which contains all the phase singularities that are associated with the optical vortices. These three functions are related to each other and are in constant interaction during propagation. By perturbing any one of these functions one destroys the equilibrium and thereby sets up a non-equilibrium homogeneous vortex field.[35] The perturbed field will evolve toward equilibrium again during subsequent propagation.

This is not the only example of a non-equilibrium homogeneous case. Another example, is where a plane wave undergoes scintillation. In this case the initial optical field contains no vortices, but if the scintillation is strong enough vortices do eventually appear. This is an important case, associated with the propagation of light through a random medium.

Here we only consider the case of a phase-corrected speckle field, where the continuous phase is removed, leaving only the amplitude and the phase singularities in the initial optical field.[36] The free-space propagation of this non-equilibrium homogeneous field has been simulated numerically to determine the evolution of the vortex density over logarithmic distances. The propagation was simulated with a Fourier optics based numerical implementation of scalar diffraction theory.[37,38]

The input speckle field, which lies in a plane perpendicular to the propagation direction, is a sampled complex-valued function consisting of 512×512 samples and it has periodic boundary conditions to avoid edge effects and aliasing. For each step the total number of vortices that are located inside the 512×512 sample window is determined as a function of the logarithmic propagation distance. The simulation was repeated several times for different initial speckle fields. The results were averaged and the standard deviations were computed for every point along the propagation distance. The wavelength and the coherence length of the optical field were chosen to ensure that the field is well within the paraxial limit.

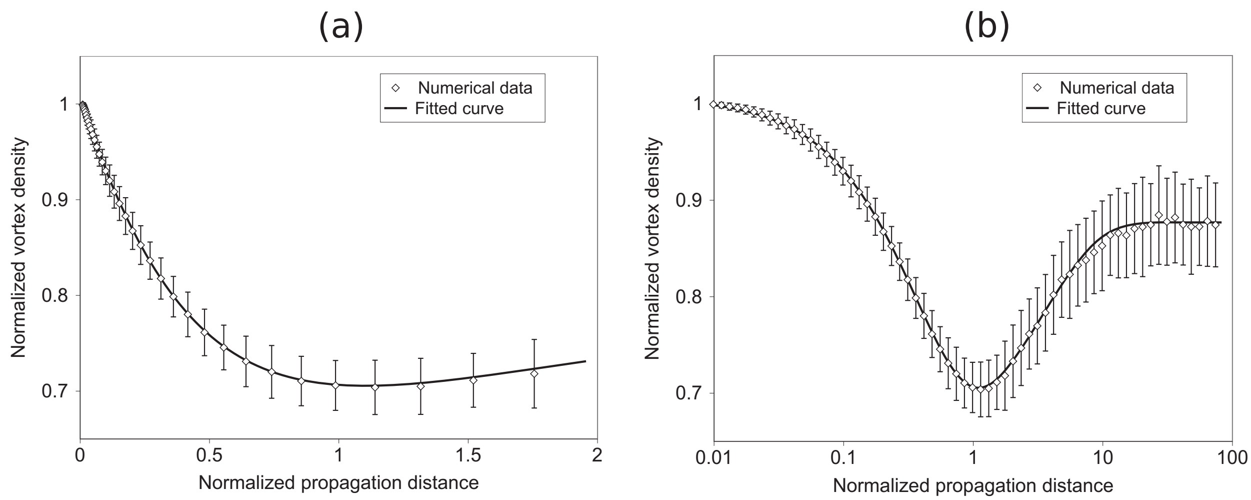

Figure 1. Normalized optical vortex density for a phase corrected speckle field, shown as a function of (a) linear normalized propagation distance, as well as (b) logarithmic normalized propagation distance. The diamonds represent numerical data, averaged over more than a hundred different simulations. The error bars represent standard deviations. The solid curve is a fitted curve through these data points.

The normalized vortex density is shown in Fig. 1 as a function of propagation distance. It reveals an unexpected behavior. After the continuous phase has been removed at $z = 0$, the vortex density drops drastically to a minimum value of about 70% of the initial vortex density. The vortex density then rises again at a rate that is about an order of magnitude slower than the rate at which it dropped. It eventually reaches an equilibrium value of about 87% of the initial vortex density.

The shape of the vortex density curve is surprising in that it does not reveal a simple exponential decay process, which can be described by a first order differential equation. Instead, the observed process follows a curve that cannot be produced by a single first order differential equation, because the slope of the function is not directly related to that function value. The curve therefore requires a second order differential equation or a set of coupled first order differential equations, such as,

$$\frac{\partial V(z)}{\partial z} + f_1(z)V(z) + f_2(z)N(z) + A_1 = 0 \quad (3)$$

$$\frac{\partial N(z)}{\partial z} + f_1(z)N(z) + f_2(z)V(z) + A_2 = 0, \quad (4)$$

from which a second order differential equation can be derived by eliminating $N(z)$. The solution of this set of differential equations with an appropriate choice of $f_1(z)$ and $f_2(z)$ can be made to fit the curve in Fig. 1 very well. The question is, what is the unknown field $N(z)$? This unknown field cannot be the topological charge density, because the latter is identically zero in this case. The unknown field would therefore have to be derived from the additional degrees of freedom that is present in the optical field, but not represented in terms of the vortex densities.

4. INHOMOGENEOUS FIELDS WITH ONE-DIMENSIONAL VARIATION

An inhomogeneous stochastic vortex field can be produced in a brute force manner by using a diffractive optical element or a spatial light modulator that inprints a phase pattern with several inhomogeneously distributed phase singularities on an illuminating plane wave. The initial amplitude of such a beam would be uniform and the optical vortices thus produced would be point vortices. The implied vortex profile function[39] would have an effect on the subsequent evolution of the beam and the vortex density.

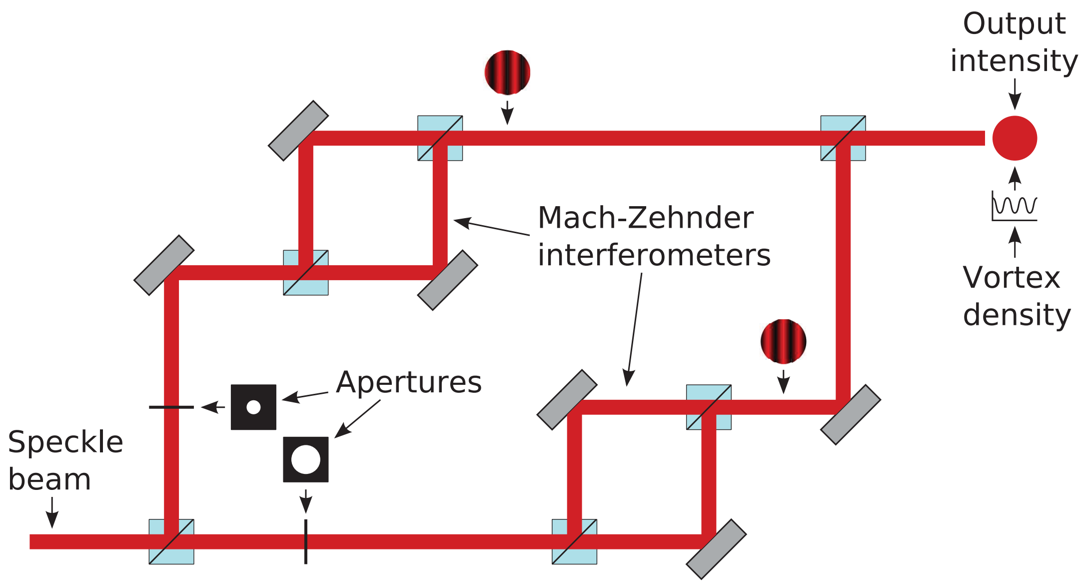

Figure 2. Setup to produce inhomogeneous initial vortex distribution. Consists of two Mach-Zehnder interferometers that are used to produce interference patterns in speckle fields with different speckle sizes, which are then combined to give a sinosoidal variation in the vortex density.

A more elegant way to generate an inhomogeneous initial vortex distribution is with the aid of the setup shown in Fig. 2. The input is a speckle field, which is divided by a 50/50 beam splitter. Each of the two resulting beams is then spatially filtered (only the circular apertures are shown) to produce speckles of different sizes in the two beams. The beams are sent through Mach-Zehnder interferometers to produce sinusoidal interference patterns. They are then recombined in such a way that the dark bands in the interference pattern of the one beam overlaps with the bright bands of the other beam's interference pattern. Because the vortex densities in the two beams are different, the resulting combined beam will have a sinusoidally varying vortex density, with a constant average intensity.

The propagation of such a stochastic vortex field has been simulated numerically. The resulting evolution of the vortex density is shown by the discrete points in Fig. 3 as a function of logarithmic propagation distance for three Fourier coefficients: the constant background level, the fundamental spatial frequency and the first harmonic. The constant background level, which represents the overall vortex density, remains more or less constant apart from some trancient variations. The fundamental spatial frequency component gives the amplitude of the sinusoidal variations in the initial vortex density. These sinusoidal variations die out during propagation as expected, but the curve contains some curious oscillations, co-located with the trancients in the background. The first harmonic start out being zero, since the interference does not initially have higher harmonic components.

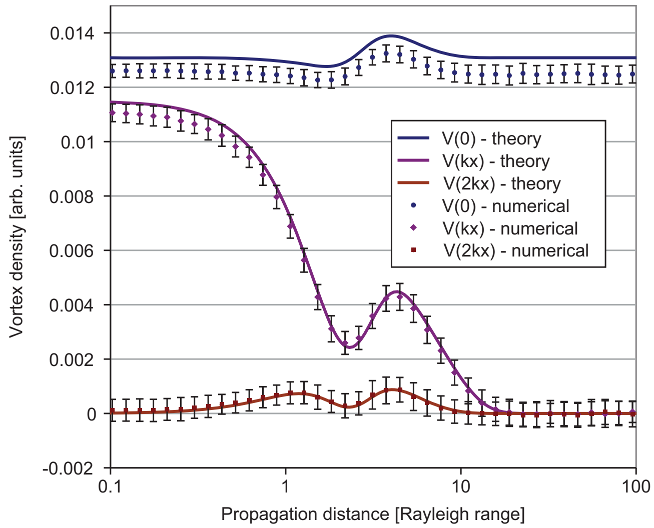

Figure 3. The evolution of an inhomogeneous optical vortex density, shown in terms of three Fourier coefficients of the optical vortex density function: the constant background level, the fundamental spatial frequency and the first harmonic. These results were obtained from the average over several simulations with the error bars indicating the standard deviations.

Yet, during propagation, the first harmonic displays some variations that are again co-located with those in the other two components. The presence of the variations in the first harmonic gives a clear indication of a non-linearity in the dynamics.

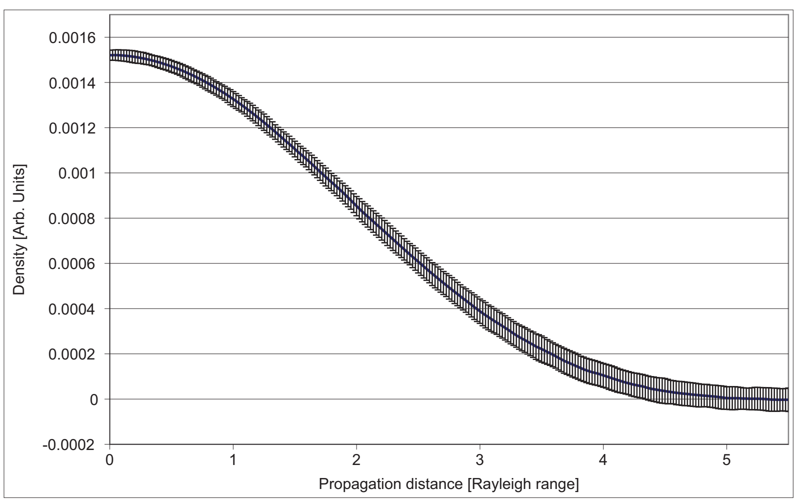

Figure 4. The evolution of an inhomogeneous topological charge density, shown in terms of the Fourier coefficients for the fundamental spatial frequency. The discrete points were obtained from the average over several simulations with the error bars indicating the standard deviations. The solid lines are analytical predictions based on statistical optical calculations.

Because the initial vortex density is produced through the interference and incoherent combination of speckle beams, one can assume Gaussian statistics and therefore employ statistical optical methods to compute the expected behavior of the vortex density. The results of these calculations* are given by the solid curves in Fig. 3. Apart from a slight offset in the magnitude of the larger densities, these analytic curves precisely follow the numerical curves. The offset ($\sim 4\%$) can be understood as an indication of the less than 100% efficiency of the vortex extraction process that is used in the numerical simulations. The statistical optics calculations and the numerical simulations are therefore in good agreement with each other.

One can also investigate the behaviour of an initially inhomogeneous topological charge density, using a setup similar to the one shown in Fig. 2. The only differences are that one keeps the size of the speckles in the two beams the same, and when the two beams are recombined one introduces a tilt along the direction parallel to the interference fringes. This will cause a relative tilt in the wavefront for adjacent bands, which in turn will result in optical vortices with predominantly the same topological charge along the overlapping regions. The result is a sinusoidally varying topological charge density.

The evolution of such a topologically charged density, as determined through numerical simulations, is shown in Fig. 4. Only the Fourier coefficient for the fundamental spatial frequency is shown. The overall topological charge density and the Fourier coefficient for the harmonics are zero. Similar result are obtined for the brute force case where the one-dimensional inhomogenouse topological charge density is introduced with the aid of direct phase modulation. It was found that for this case the rate of decay is proportional to the square of the spatial frequency.[40] This behavior is therefore governed by a simple diffusion equation,

$$\partial_z D(x,z) - \kappa(z)\nabla^2 D(x,z) = 0 \tag{5}$$

where the diffusion coefficient depends linearly on the propagation distance, $\kappa(z) = \kappa_0 z$. Such a diffusion is caused by the random motions of the vortices during propagation.

5. INHOMOGENEOUS FIELDS WITH TWO-DIMENSIONAL VARIATION

When the initial inhomogeneous vortex density and/or topological charge density have variations in two dimensions one finds that more mechanisms come into play during the propagation of the vortex field. In addition to the diffusion mechanism mentioned in the previous section one now also finds drift mechanisms. The evidence for that lies in the non-linear behaviour of the topological charge density, because these drift mechanisms imply a non-linear behaviour.

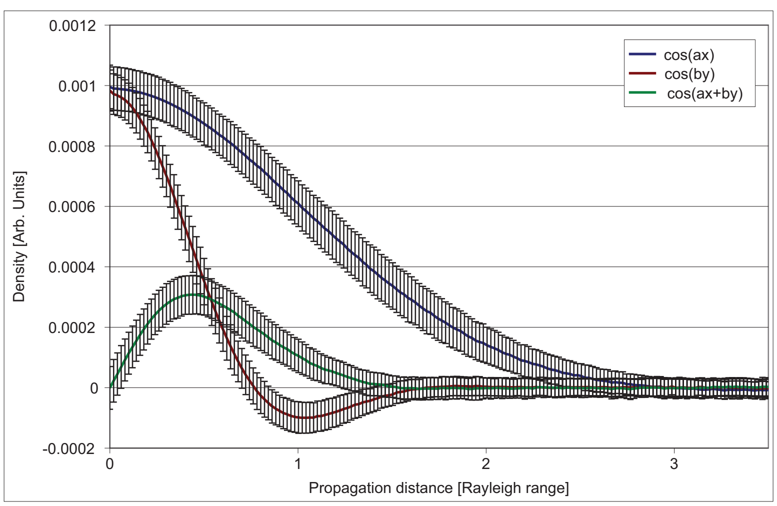

Figure 5. The evolution of an inhomogeneous topological charge density, shown in terms of thee Fourier coefficients for the small, large and mixed frequency components of the topological charge density function. These results were obtained from the average over several simulations with the error bars indicating the standard deviations.

One can produce an initial vortex field with a two-dimensional topological charge density variations using direct phase modulation. The subsequent propagation of this initial vortex field has been simulated numerical. In Fig. 5 we show the evolution of three Fourier components for such a case. In this case the initial topological charge density is given by the product of two cosine functions, $\cos(ax)\cos(by)$, where x and y are the transverse coordinates along orthogonal directions and a and b represent the angular spatial frequencies, such that $a \neq b$. The three curves in Fig. 5 represent the Fourier coefficients for $\cos(ax)$, $\cos(by)$ and $\cos(ax + by)$, which are respectively the small, large and mixed frequency components. The small frequency component decays in a way very similar to the one-dimensional case. The large frequency component decays faster, as one would expect, but it is driven negative before it finally decays to zero. The mixed component, which is not initially present, grows from zero to reach a peak and then decays again to zero — a clear indication of nonlinear dynamics.

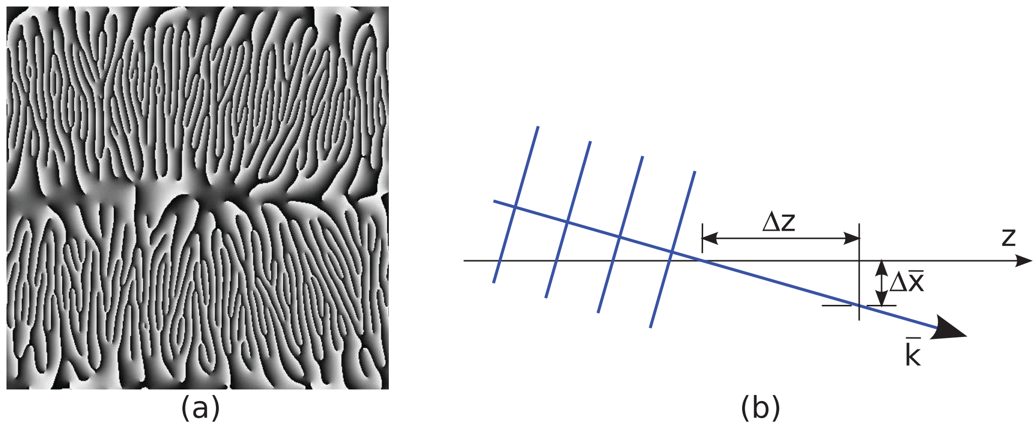

Figure 6. Phase drift mechanism: An to inhomogeneous vortex field (a) contains regions of tilted phase, which gives rise to a lateral drift (b) in the wavefront.

The nonlinear behaviour is produced by a drift mechanism. Such a mechanism can be produced by the phase variation that is imposed by the inhomogeneous vortex distribution as shown in Fig. 6(a). The phase variation between regions of opposite topological charge would on average have a tilted wavefront, which would cause the wave in that region to drift sideways during propagation. If the topological charge density has a one-dimensional variation the lateral drift would be along a homogeneous direction with now change to the overall distribution function. On the other hand, if the topological charge density has a two-dimensional variation the lateral drift can cause a change in the shape of the overall distribution function, which implies a nonlinear mechanism.

6. SUMMARY

Numerical results obtained from simulations of the evolution of initial stochastic vortex fields are shown, together with some analytical results. The cases that are considered include homogeneous non-equilibrium stochastic vortex fields, one-dimensional inhomogeneous stochastic vortex fields and two-dimensional inhomogeneous stochastic vortex fields. We find that some cases (such as the one-dimensional inhomogeneous cases) give results that can be described by linear diffusion processes, while other cases (such as two-dimensional inhomogeneous cases) also required non-linear drift mechansisms in the field. If the initial optical field obeys Gaussian statistics one can use a statistical optics approach to compute analytical curves for the evolution of the vortex distributions. We also find that a non-equilibrium homogeneous initial vortex field does not give a simple decay process to restore equilibrium. More complicated dynamics are involved, which requires deeper investigations.

REFERENCES

[1] Nye, J. F. and Berry, M. V., "Dislocations in wave trains," *Proc. R. Soc. Lond. A* **336**, 165–190 (1974).

[2] Dennis, M. R., O'Holleran, K., and Padgett, M. J., "Singular Optics: Optical Vortices and Polarization Singularities," *Progress in Optics* **53**, 293–363 (2009).

[3] Allen, L., Beijerbergen, M. W., Spreeuw, R. J. C., and Woerdman, J. P., "Orbital angular momentum of light and the transformation of laguerre-gaussian laser modes," *Phys. Rev. A* **45**, 8185–8189 (1992).

[4] Soskin, M. S., Gorshkov, V. N., Vasnetsov, M. V., Malos, J. T., and Heckenberg, N. R., "Topological charge and angular momentum of light beams carrying optical vortices," *Phys. Rev. A* **56**, 4064–4075 (1997).

[5] Berry, M. V., "Paraxial beam of spinning light," *Proc. SPIE* **3487**, 6–11 (1998).

[6] Indebetouw, G., "Optical vortices and their propagation," *J. Mod. Opt.* **40**, 73–87 (1993).

[7] Roux, F. S., "Dynamical behavior of optical vortices," *J. Opt. Soc. Am. B* **12**, 1215–1221 (1995).

[8] Rozas, D., Law, C. T., and Swartzlander Jr., G. A., "Propagation dynamics of optical vortices," *J. Opt. Soc. Am. B* **14**, 3054–3065 (1997).

[9] Freund, I., "Optical vortex trajectories," *Opt. Commun.* **181**, 19–33 (2000).

[10] Freund, I., "Critical point explosions in two-dimensional wave fields," *Opt. Commun.* **159**, 99–117 (1999).

[11] Freund, I. and Kessler, D., "Critical point trajectory bundles in singular wave fields," *Opt. Commun.* **187**, 71–90 (2001).

[12] Molina-Terriza, G., Wright, E. M., and Torner, L., "Propagation and control of noncanonical optical vortices," *Opt. Lett.* **26**, 163–165 (2001).

[13] Schwarz, U. T., Sogomonian, S., and Maier, M., "Propagation dynamics of phase dislocations embedded in a bessel light beam," *Opt. Commun.* **208**, 255–262 (2002).

[14] Singh, R. P. and Chowdhury, R., "Trajectory of an optical vortex: canonical vs. noncanonical," *Opt. Commun.* **215**, 231–237 (2003).

[15] Roux, F. S., "Canonical vortex dipole dynamics," *J. Opt. Soc. Am. B* **21**, 655–663 (2004).

[16] Roux, F. S., "Coupling of noncanonical optical vortices," *J. Opt. Soc. Am. B* **21**, 664–670 (2004).

[17] Flossmann, F., Schwarz, U. T., and Maier, M., "Propagation dynamics of optical vortices in laguerre-gaussian beams," *Opt. Commun.* **250**, 218–230 (2005).

[18] Kim, G.-H., Lee, H. J., Kim, J.-U., and Suk, H., "Propagation dynamics of optical vortices with anisotropic phase profiles," *J. Opt. Soc. Am. B* **20**, 351–359 (2003).

[19] Roux, F. S., "Topological charge inversion in polynomial astigmatic gaussian beams," *Opt. Commun.* **281**, 4205–4210 (2008).

[20] Schechner, Y. Y. and Shamir, J., "Parameterization and orbital angular momentum of anisotropic dislocations," *J. Opt. Soc. Am. A* **13**, 967–973 (1996).

[21] Roux, F. S., "Distribution of angular momentum and vortex morphology in optical beams," *Opt. Commun.* **242**, 45–55 (2004).

[22] Roux, F. S., "Spatial evolution of the morphology of an optical vortex dipole," *Opt. Commun.* **236**, 433–440 (2004).

[23] Dana, I. and Freund, I., "Vortex-lattice wave fields," *Opt. Commun.* .

[24] Jenkins, R., Banerji, J., and Davies, A., "The generation of optical vortices and shape preserving vortex arrays in hollow multimode waveguides," *J. Opt. A: Pure Appl. Opt.* **3**, 527–532 (2001).

[25] Dreischuh, A., Chervenkov, S., Neshev, D., Paulus, G., and Walther, H., "Generation of lattice structures of optical vortices," *J. Opt. Soc. Am. B* **19**, 550–556 (2002).

[26] Courtial, J., Zambrini, R., Dennis, M., and Vasnetsov, M., "Angular momentum of optical vortex arrays," *Opt. Express* **14**, 938–949 (2006).

[27] Berry, M. V., "Disruption of wavefronts: statistics of dislocations in incoherent gaussian random waves," *J. Phys. A: Math. Gen.* **11**, 27–37 (1978).

[28] Freund, I., Shvartsman, N., and Freilikher, V., "Optical dislocation networks in highly random media," *Opt. Commun.* **101**, 247–264 (1993).

[29] Freund, I., "Optical vortices in gaussian random wave fields: statistical probability densities," *J. Opt. Soc. Am. A* **11**, 1644–1652 (1994).

[30] Shvartsman, N. and Freund, I., "Vortices in random wave fields: nearest neighbor anticorrelations," *Phys. Rev. Lett.* **72**, 1008–1011 (1994).

[31] Berry, M. V. and Dennis, M. R., "Phase singularities in isotropic random waves," *Proc. R. Soc. Lond. A* **456**, 2059–2079 (2000).

[32] Wang, W., Hanson, S. G., Miyamoto, Y., and Takeda, M., "Experimental investigation of local properties and statistics of optical vortices in random wave fields," *Phys. Rev. Lett.* **94**, 103902 (2005).

[33] O'Holleran, K., Dennis, M. R., Flossmann, F., and Padgett, M. J., "Fractality of light's darkness," *Phys. Rev. Lett.* **100** (2008).

[34] Goodman, J. W., [*Statistical Optics*], John-Wiley & Sons, Inc., New York (1985).

[35] Chen, M. and Roux, F. S., "Influence of the least-squares phase on optical vortices in strongly scintillated beams," *Phys. Rev. A* **80**, 013824 (2009).

[36] Chen, M. and Roux, F. S., "Evolution of the scintillation index and the optical vortex density in speckle fields after removal of the least-squares phase," *J. Opt. Soc. Am. A* **27**, 2138–2143 (2010).

[37] Devaney, A. J. and Sherman, G. C., "Plane-wave representations for scalar wave fields," *SIAM Rev.* **15**, 765–786 (1973).

[38] Goodman, J. W., [*Introduction to Fourier Optics, 2nd Ed.*], McGraw-Hill, New York (1996).

[39] Roux, F. S., "Optical vortex density limitation," *Opt. Commun.* **223**, 31–37 (2003).

[40] Roux, F. S., "Lateral diffusion of the topological charge density in stochastic optical fields," *Opt. Commun.* **283**, 4855–4858 (2010).