Lateral Diffusion of the Topological Charge Density in Stochastic Optical Fields

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Abstract

Stochastic (i.e. random and quasi-random) optical fields may contain distributions of optical vortices that are represented by non-uniform topological charge densities. Numerical simulations are used to investigate the evolution under free-space propagation of topological charge densities that are inhomogeneous along one dimension. It is shown that this evolution is described by a diffusion process that has a diffusion parameter which depends on the propagation distance.

Keywords: optical vortex, singular optics, stochastic optical field, topological charge density, diffusion equation

1. Introduction

The spatial information in an optical beam is distorted when the beam propagates through a turbulent medium. The slight variations in the refractive index, caused by the turbulence, introduce random phase modulations to the optical beam. Over a distance the accumulated phase modulations eventually cause distortions in the optical beam that manifest as intensity fluctuations, called scintillation.

The distortions in such a scintillated beam can, to some extent, be corrected with an adaptive optics system [1, 2], which measures the continuous phase distortions and then removes them with a continuous deformable mirror. The problem with this approach comes in with strong scintillation, when the phase distortions are severe enough to give rise to the spontaneous generation of optical vortices [3]. Before removing the continuous phase distortions in strongly scintillated beams, one first needs to remove the optical vortices.

An optical vortex [4, 5] is a singularity (branch point) that can exist in the phase function at points on the cross-section of strongly scintillated beams, causing the intensity at these points to vanish. These points are actually lines along the direction of propagation and around these lines the wavefront has a helical shape. The handedness of the helical wavefront is referred to as the topological charge of the optical vortex and is represented by the signed integers ± 1. (Topological charges of larger signed integer values can in principle also appear, but they are unstable in scintillated beams and generally decay into vortices with topological charges of ± 1.) Vortices in an optical field can be annihilated and created in oppositely charged pairs (vortex dipoles).

To address the challenge of removing unwanted optical vortices it is necessary to understand the collective behavior of optical vortices in strongly scintillated beams, as well as in other optical vortex fields. As a starting point one can consider optical speckle fields, which are better understood. When a coherent optical beam is scattered from a rough surface the scattered light forms a speckle field. The phase function of such a speckle field contains numerous phase singularities that propagate along with the field as optical vortices. A speckle field can therefore also be represented as a random optical vortex field. The properties of optical vortices in random optical fields (speckle fields) have been studied extensively [3, 6, 7, 8, 9]. It is known that the vortex density in such a speckle field is inversely proportional to the coherence area of the beam [6]. It is also known that the topological charge in these beams is distributed in such a way that nearest neighbors tend to have opposite topological charges [8], with the result that the net topological charge over any area of such a beam is minimized. The vortices in such a field are constantly being annihilated and created, but the average number of vortices in the field remains constant during propagation. One can therefore consider the random vortex distribution in a speckle field as a system in equilibrium — the rate of vortex dipole creation is balanced by the rate of vortex dipole annihilation.

On the other hand, a strong scintillation process causes the number of vortices in the beam to increase constantly. In other words, the rate of dipole creation exceeds the rate of dipole annihilation. For this reason a strongly scintillated beam does not represent a system in equilibrium while it is propagating through a turbulent medium. An optical beam that has been scintillated by a turbulent medium can however reach a state of equilibrium if it is allowed to propagate through a subsequent medium without turbulence. In such a case the average vortex density would reach a fixed value that is maintained during further free-space propagation, which implies that the equilibrium is restored.

For a more comprehensive understanding of the statistical behaviour of optical vortices one also needs to consider quasi-random fields where the vortex distributions are not spatially uniform. For this purpose one needs to investigate inhomogeneous vortex distribution, unlike the homogeneous distribu-

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tions that are obtained in speckle fields and scintillated beams. Very little is known about the evolution of inhomogeneous vortex distributions in quasi-random optical vortex fields. Random and quasi-random optical field are here collectively referred to as stochastic optical fields. The optical vortices in such stochastic optical beams can be represented by statistical number densities for each of the two topological charges. These number densities are not only functions of the propagation distance, but also depend on the lateral coordinate due to there inhomogeneous nature. They can be used to define the topological charge density of the optical vortex field. It is given by the difference between the positive vortex number density and the negative vortex number density.

In this paper, we develop a differential equation to describe the evolution during propagation of one-dimensional non-uniform topological charge densities in stochastic optical fields, that are generated by direct phase modulation. This equation is a diffusion equation. (We deliberately exclude drift effects. See Section 4.) The diffusion coefficient is investigated with the aid of numerical simulations of the propagation of such vortex fields. The dominant Fourier components of the numerically simulated evolution of such topological charge densities are extracted to confirm the diffusive behavior.

The current work builds on previous work [10, 11]. In Ref. [10] an investigation was made into the restoration scale of the optical vortex density in an inhomogeneous vortex field, however the work there did not address the underlying dynamics, apart from a qualitative discussion based on a vortex plasma model. The work of Ref. [11] considered the evolution of the vortex density in a homogenous vortex field after the equilibrium is destroyed by removing the continuous phase, without providing any quantitative model for the underlying dynamics.

The current work considers the evolution of the topological charge density (not the vortex density) in an inhomogeneous stochastic vortex field and it specifically addresses the theoretical aspects of the underlying dynamics that has not been considered before.

2. Topological charge density and vortex density

Recently, an optical vortex plasma model was proposed for the evolution of optical vortex fields during propagation [10]. In this model, positive and negative vortices are represented by their number densities, $n_p(x,y,z)$ and $n_n(x,y,z)$, respectively. The hypothetical neutral bound state number density, which was also introduced in Ref. [10] will not be considered here. The vortex plasma model has been used [11] to provide a qualitative explanation for the evolution of homogeneous vortex fields that are perturbed away from equilibrium. Due to the finite extent of an optical beam, as well as random variations that may exist in the beam, the assumption of a homogeneous field is not in general valid. It is therefore necessary to include spatial variations in the densities of the optical vortices. To understand the collective behavior of the optical vortices, one needs to understand how the spatial variations affect local changes in the densities as a function of the propagation distance.

By implication, topological charge is topologically conserved — the net flow of topological charge into a closed volume is zero. This suggests that one can associate a conserved current with each of the densities. These densities are conserved up to the difference between the number of creations and annihilations. The conserved currents therefore obey the following equations,

$$\partial_z n_p(x,y,z) + \nabla \cdot \mathbf{J}_p(x,y,z) = C - \mathcal{A} \tag{1}$$
$$\partial_z n_n(x,y,z) + \nabla \cdot \mathbf{J}_n(x,y,z) = C - \mathcal{A}, \tag{2}$$

where $\mathbf{J}_p(x,y,z)$ and $\mathbf{J}_n(x,y,z)$ are transverse currents that are associated with $n_p(x,y,z)$ and $n_n(x,y,z)$, respectively; C and $\mathcal{A}$ respectively represent the creation and annihilation events, and $\nabla = \partial_x\hat{x} + \partial_y\hat{y}$.

One can use the conservation equations to eliminate the dependence on the creations and annihilations. This is done by subtracting Eq. (2) from Eq. (1) and work instead with the topological charge density, $D(x,y,z) = n_p(x,y,z) - n_n(x,y,z)$. The resulting conservation equation for topological charge density is,

$$\partial_z D(x,y,z) + \nabla \cdot \mathbf{J}_D(x,y,z) = 0, \tag{3}$$

where $\mathbf{J}_D(x,y,z) = \mathbf{J}_p(x,y,z) - \mathbf{J}_n(x,y,z)$, which is the transverse topological charge current, associated with $D(x,y,z)$. The conservation equation in Eq. (3) is a direct mathematical expression of the conservation of topological charge.

One can also consider the vortex density, which is defined as the sum: $V(x,y,z) = n_p(x,y,z) + n_n(x,y,z)$, giving a conservation equation that still contains the creations and annihilations,

$$\partial_z V(x,y,z) + \nabla \cdot \mathbf{J}_V(x,y,z) = 2\,(C - \mathcal{A})\,, \tag{4}$$

where the vortex current $\mathbf{J}_V(x,y,z) = \mathbf{J}_p(x,y,z) + \mathbf{J}_n(x,y,z)$ is associated with $V(x,y,z)$.

Only the topological charge density is further considered in this paper.

3. Numerical simulations

In the next section results are presented that were obtained from numerical simulations that simulate the paraxial scalar propagation of a monochromatic optical field. These simulations were used to investigate the evolution of distributions of optical vortices in these optical fields.

The numerical beam propagation algorithm that is used here, is an implementation of scalar diffraction theory based on Fourier optics [12, 13]. The procedure itself consists of three steps: first the input optical field is Fourier transformed to obtain the angular spectrum; then this angular spectrum is multiplied by the propagation phase function, which depends on the distance of propagation; finally the output field is reconstructed at the output plane by performing an inverse Fourier transform on the modified angular spectrum.

The input optical fields are sampled complex-valued functions, consisting of 512×512 samples, that represent the beam cross-section in the plane where the optical vortex distribution

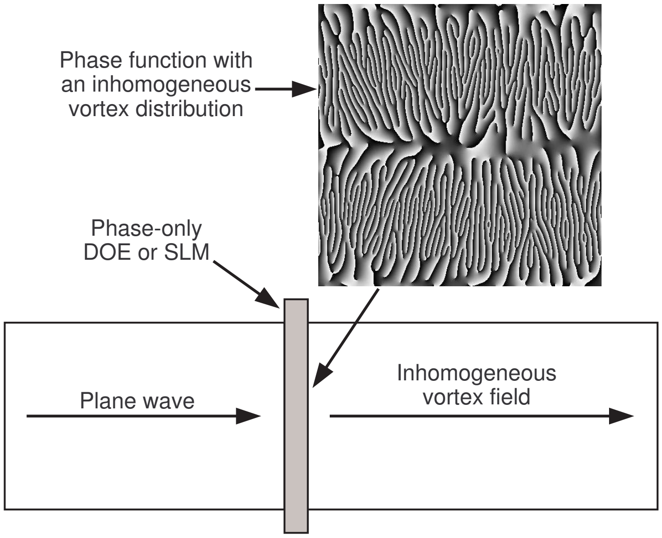

Figure 1: The physical setup that is being simulated, showing an example of a typical phase function that is used in the direct phase modulation process.

is defined. Physically this optical vortex distribution is produced by direct phase modulation that can either be accomplished by a spatial light modulator (SLM) or a phase-only diffractive optical element (DOE). The simulated setup is illustrated in Fig. 1. A plane wave illuminates a phase-only DOE or an SLM (shown as a transmissive device) that only modulates the phase of the plane wave. The resulting optical field after the phase modulation, would initially have a uniform amplitude, just like the plane wave that was incident on the DOE or SLM. Therefore, the complex-valued input function in the simulation also have a uniform amplitude (intensity), with a phase function that typically looks like the inset shown in Fig. 1. Note that the stochastic aspect of the field is introduced by the quasi-random locations of the phase singularities in the phase function with which the incident plane wave is modulated. Although it is quasi-random, the distribution of the topological charges of the vortices is given by some predetermined function, such as a cosines function.

To avoid edge effects and aliasing the complex-valued functions are always produced with periodic boundary conditions. In other words, the opposite edges of such a function match each other continuously, so that the function could be used to tile the infinite two-dimensional plane to produce a continuous function (apart from the phase singularities). As a result the optical field does not expand during propagation.

The numerical procedure then propagates the initial function through free-space over progressively larger distances. The propagation distance is incremented in steps to determine the effect of the propagation distance on the evolution of the topological charge distribution. For each step the topological charge density is obtained by computing the location and topological charge for every vortex in the beam. This is done through a numerical implementation of the curl of the gradient of the phase function,

$$D(x, y) = \frac{1}{2\pi}\nabla_T \times \nabla_T \theta(x, y), \tag{5}$$

where ∇_T is the two-dimensional gradient operator on the xy-plane. The resulting topological charge density $D(x, y)$ is a two-dimensional sampled function that is zero everywhere except at the locations of the vortices, where it is equal to ± 1, indicating the topological charge of the vortex. The Fourier transform of this topological charge density is used to extract the amplitudes of the pertinent Fourier components of the topological charge density. All Fourier transforms are performed numerically using a Fast Fourier Transform procedure.

In all the numerical analyses the wavelength was chosen small enough to stay within the paraxial limit and avoid the limitations that exist for topological charge densities [14].

4. Lateral motion of optical vortices during propagation

To determine the dependence of the currents on the number densities, one can investigate the transverse motion of individual optical vortices in the beam. We are particularly interested in how the motion of a vortex is influenced during propagation by properties of the beam in the vicinity of the vortex. Fluctuations in the phase and amplitude in the region surrounding a vortex, would in general influence the motion of the vortex [15, 16]. These fluctuations can be caused by other vortices in the region or by the continuous variations in the beam itself.

Inspired by the successes of statistical physics, one can assume that changes in the vortex number densities on the transverse plane can either be produced by the diffusion of the vortices or due to a drift caused by some ‘force’. One can therefore replace the divergence of the current in Eqs. (1-4) by the sum of a diffusion term and a drift term.

In this paper we restrict ourselves to the diffusion effect. One can show that the mechanisms for lateral vortex drift [15, 16] are always perpendicular to the variations in the vortex distributions. As a result the drift mechanisms vanish when the distributions only have a one-dimensional dependence. Hence, we only consider vortex distributions with one-dimensional variations.

It is reasonable to expect that the diffusion process is caused by random motions of the optical vortices, which can be modelled by a random walk process. The detailed mechanism for these random motions then becomes unimportant. The change in the distribution of vortices as a function of propagation distance should then be proportional to the Laplacian of the distribution. For the topological charge density $D(x, y, z)$ this is given by,

$$\partial_z D(x, y, z) - \kappa\nabla^2 D(x, y, z) = 0, \tag{6}$$

where κ is an as yet unknown diffusion coefficient. This is simply the diffusion (or heat) equation for two spatial dimensions, where the propagation distance z takes over the role of the time dimension. Note that Eq. (6) has the same form as Eq. (3); the divergence of the current is now replaced by the Laplacian term.

The solutions of the diffusion equation are of the form,

$$D(\mathbf{x}, z) = \exp(-\kappa|\mathbf{a}|^2 z)\cos(\mathbf{a}\cdot\mathbf{x}), \tag{7}$$

where **x** represents the position vector on the two-dimensional transverse plane and **a** denotes a two-dimensional spatial frequency vector. The distribution decays exponentially at a rate that increases for distributions with larger spatial frequencies.

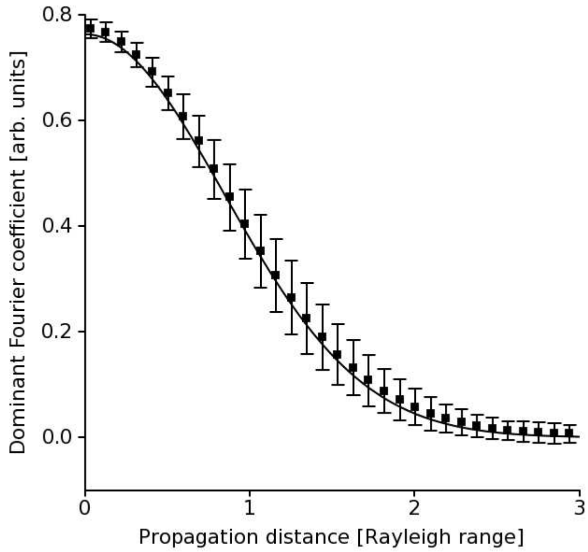

Figure 2: Decay curve as a function of the propagation distance measured in Rayleigh range ($\pi d^2/\lambda$, where d is the average separation distance between vortices in the initial vortex distribution). The numerical results, obtained from the propagation of an initial cosines topological charge density, are shown by squares. The error bars denote the standard deviations. The solid line shows a Gaussian function fitted through the numerical data.

Several different input function were generated with different sets of parameters. All these input function contained topological charge distributions with a one-dimensional sinusoidal spatial dependence. This sinusoidal dependence represented the dominant Fourier components in the Fourier transform of the topological charge density. Using the procedure explained in Sec. 3, we produced curves for the decay of the dominant Fourier component as a function of the propagation distance. A subset of these results were used for the averaged decay curve shown in Fig. 2. For this subset the initial distributions all contained exactly 800 vortices each (within the 512×512 sample window), distributed in such a manner that the topological charge density forms a one-dimensional cosine along the x-direction, with exactly three periods fitting into the 512×512 sample window. A wavelength of 1.667 sample spacings was used for this case.

In Fig. 2 the squares represent the average (normalized) amplitude of the dominant Fourier component as a function of the propagation distance, measured in Rayleigh range ($\pi d^2/\lambda$, where d is the average separation distance between vortices in the initial vortex distribution), and the error bars represent the standard deviations. The solid line is a least-squares fit with a Gaussian function. It is apparent from the curve in Fig. 2 that, although these distributions do decay, their dependence on the propagation distance follows a Gaussian shape as opposed to the exponential decay indicated in Eq. (7).

One can understand the shape of the decay curve as a manifestation of a broken shift invariance. Although the solution in Eq. (7) is shift invariant in z, in a practical situation (as simulated by our numerical procedure) the shift invariance is explicitly broken by the fact that the beam is generated by a direct phase modulation (e.g. by a DOE located at a specific point along the propagation direction). The subsequent evolution of the distribution after the DOE can therefore not be expected to be shift invariant.

Another observation that explains the shape of the decay curve is the fact that the same decay curve is seen in the reverse propagation direction, when the beam is numerically propagated backward. This symmetric behavior should manifest as a mirror symmetry of the diffusion equation along z. When $z \to -z$, which implies that $D \to -D$, the same differential equation should be recovered. The expression in Eq. (6) does not have such a symmetry. However, if the diffusion constant κ is replaced by an anti-symmetric function of z, the equation will have the required mirror symmetry.

One can modify the diffusion equation in Eq. (6) to incorporate the explicit dependence on the propagation distance. For this purpose we assume that the diffusion coefficient κ is a function of the propagation distance. The simplest function that is in agreement with the simulation results, is $\kappa(z) = (z - z_0)\kappa_0$, where z_0 denotes the location of the DOE and κ_0 is a dimensionless constant.

If we redefine the z-axis so that the DOE is located at $z = 0$, the modified diffusion equation is given by,

$$\partial_z D(x, y, z) - \kappa_0 z \nabla^2 D(x, y, z) = 0, \tag{8}$$

and its solutions are of the form,

$$D(x, z) = \exp\left(-\frac{\kappa_0}{2}|\mathbf{a}|^2 z^2\right)\cos(\mathbf{a} \cdot \mathbf{x}), \tag{9}$$

which is in agreement with the numerical results shown in Fig. 2.

By varying the wavelength, initial vortex density and spatial frequency of the topological charge density in our numerical simulations, we could verify that the dependence of the parameters as presented in Eq. (9) is correct. We could also determine how the constant κ_0 depends on these parameter. We thus obtained the empirical expression $\kappa_0 = \zeta\lambda^2/d^2$, where ζ is a constant of order 1, λ is the wavelength and d is the average separation distance between vortices in the initial vortex distribution. The value of ζ that was obtained from the numerical data of 350 simulations, using a Monte-Carlo approach, is, $\zeta = 0.32 \pm 0.09$, which is consistent with a value of $1/\pi$.

5. Summary and conclusions

A diffusion equation is proposed that describes the evolution of stochastic optical fields with one-dimensional non-uniform topological charge densities during free-space propagation. The diffusion term contains a diffusion coefficient that depends on the propagation distance. Numerical simulations are performed to investigate the shape of the decay curves. It was found that the diffusion parameter is proportional to the propagation distance. This is consistent with the symmetry requirements of the equation. When the beam profile is produced with direct phase modulation, the random motions of the vortices are not governed by a normal random walk process. Instead the size of random motions seems to depend on the distance from the plane where the direct phase modulation is introduced.

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