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ON GYROSCOPIC EFFECTS IN VIBRATING AND AXIALLY ROTATING SOLID AND ANNULAR DISCS

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Abstract

“Bryan’s effect” - that is, the effect of a vibrating pattern’s precession in the direction of inertial rotation of a vibrating ring - was discovered by G. Bryan in 1890. This effect has several applications in navigational instruments, such as cylindrical, hemispherical and planar circular disc rotational sensors. The model of a thin circular disc vibrating in its plane and subjected to inertial rotation is considered. The dynamics of the disc gyroscope are considered in terms of linear elasticity. Two models are considered: solid discs and a composite disc consisting of concentric annular discs with various boundary conditions on the inner and outer circumferences. It is assumed that the angular rate of inertial rotation of the composite disc is constant and has axial orientation. It is also assumed that this angular rate is much smaller than the lowest eigenvalue of the composite disk. Hence any centrifugal effects and quantities that are proportional to the square of the angular rate are negligible. Our model is formulated in general terms and then compared to a formulation in terms of Novozhilov-Arnold-Warburton’s theory of thin shells. The system of equations of motion of the disc is separated and transformed into a pair of wave equations in polar coordinates. A solution is obtained in terms of Bessel and Neumann functions. Various non-axisymmetric modes of the composite disc are considered and the dependence of Bryan’s effect on eigenvalues, mass densities of the composite disc, its modulii of elasticity, Poisson ratios, outer and inner radii of the disc, and for various types of boundary conditions, are investigated.

1. INTRODUCTION

The effect of a vibrating pattern’s precession in the direction of inertial rotation of a vibrating ring was discovered by G. Bryan [1] in 1890. In this paper we consider the model of a thin circular disc vibrating in its plane and subjected to inertial rotation. The main gyroscopic effect of the vibrating disc can be described as follows: The angular rate of a vibration pattern is proportional to the inertial angular rate applied to the structure. This phenomenon, known as “Bryan’s effect”, depends on the particular eigenvalue involved. This effect has several applications in navigational instruments, such as cylindrical, hemispherical and planar circular disc rotational sensors (see, for instance, [2]).

Solutions to the system of equations of motion of the composite disc are derived from

Lagrange's variational principle. The solutions are obtained in terms of Bessel and Neumann functions. Bryan's effect, for various eigenvalues, is investigated for various types of inner boundary conditions, circumferential wave numbers and Poisson ratios.

2. GYROSCOPIC EFFECTS IN PLANE BODIES

Let us consider a composite circular disc consisting of a number of concentric annular discs (Fig. 1). Suppose the composite disc is subjected to non-decaying vibrations on one of its natural modes. Assume that the composite disc is rotating at a constant inertial angular rate Ω with axis of rotation the Oz axis. It is assumed that Ω is small compared to the lowest eigenvalue of the system. Hence it is possible to neglect any centrifugal effects as well as $O\left(\Omega^2\right)$ in calculations involving any eigenvalue.

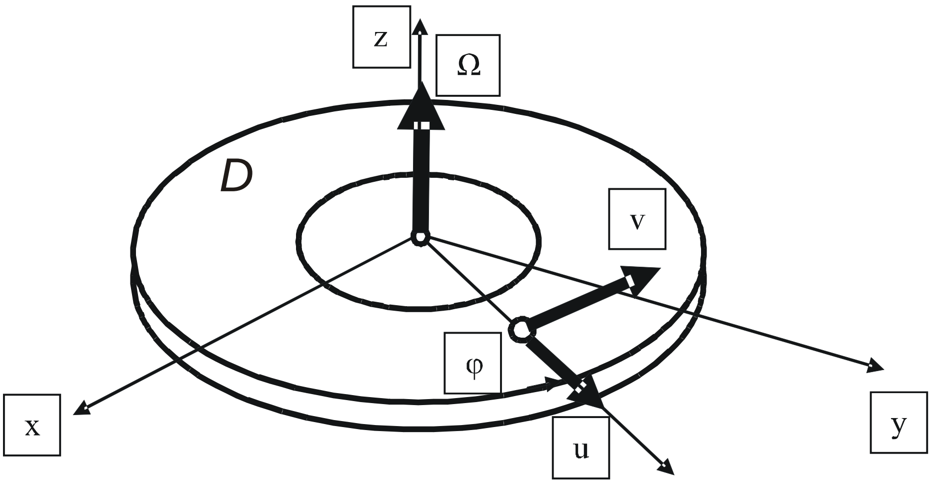

Figure 1. Coordinate system for concentric annuli

Assume that $\left(u, v, w\right)^T$ is the vector of linear displacements of an arbitrary point P, belonging to the composite disc $\left(D\right)$. The normal component of displacement is assumed to be equal to zero ($w = 0$). Consequently, the absolute linear velocity of this point is

$$\vec{V} = \begin{bmatrix} \dot{u} - \Omega v \\ \dot{v} + \Omega\left(r + u\right) \\ 0 \end{bmatrix} \tag{1}$$

where r is the distance from the centre O to point P of the body. The kinetic energy of the system of annular discs is as follows:

$$T \approx \frac{1}{2}\sum_{i=1}^{N} \rho_i h_i \int_0^{2\pi} \int_{a_{i-1}}^{a_i} \left[\left(\dot{u}_i^2 + \dot{v}_i^2\right) + 2\Omega\left(u_i \dot{v}_i - \dot{u}_i v_i\right)\right] r\, dr d\varphi \tag{2}$$

Where N is the number of concentric annular discs in the system, a_{i-1}, a_i are the inner and outer radii, ρ_i the density and h_i the thickness and of the i^{th} annulus ($i = 1, 2, \ldots, N$).

We express displacements $u_i = u_i(r,\varphi,t)$ and $v_i = v_i(r,\varphi,t)$ of the i^{th} annulus of the system as follows:

$$\begin{aligned} u_i(r,\varphi,t) &= U_i(r)\left[C(t)\cos(m\varphi) + S(t)\sin(m\varphi)\right]; \\ v_i(r,\varphi,t) &= V_i(r)\left[C(t)\sin(m\varphi) - S(t)\cos(m\varphi)\right] \end{aligned} \tag{3}$$

where $U_i = U_i(r)$, $V_i = V_i(r)$ are the eigenfunctions of the system corresponding to eigenvalues ω, which will be calculated later, and m, an integer, is the circumferential wave number. The nature of the functions $C = C(t)$ and $S = S(t)$ can be determined from Eq. (4) below.

After substituting Eq. (3) into Eq. (2) we obtain an expression for the kinetic energy of the system $T = T(\dot{C},\dot{S},C,S)$. The system of equations for the mode under consideration is:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{C}}\right) - \frac{\partial T}{\partial C} = -\omega^2 C; \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{S}}\right) - \frac{\partial T}{\partial S} = -\omega^2 S \tag{4}$$

or

$$\ddot{C} + 2\eta\Omega\dot{S} + \omega^2 C = 0; \qquad \ddot{S} - 2\eta\Omega\dot{C} + \omega^2 S = 0 \tag{5}$$

where the so-called “Bryan’s factor” η (see [3]) is given by:

$$|\eta| = \left| \frac{2 \cdot \sum_{i=1}^{N} \left\{ \rho_i h_i \cdot \int_{a_{i-1}}^{a_i} U_i(r) V_i(r) r dr \right\}}{\sum_{i=1}^{N} \left\{ \rho_i h_i \cdot \int_{a_{i-1}}^{a_i} \left[U_i^2(r) + V_i^2(r)\right] r dr \right\}} \right| \le 1 \tag{6}$$

We can interpret Bryan’s factor as follows: Let $X = C + iS$ ($i^2 = -1$). Then, neglecting $O(\Omega^2)$ terms, from Eq. (5) and considering the transformation

$$X(t) = Y(t) \cdot \exp(i\eta\Omega t) \tag{7}$$

we arrive at the approximation $\ddot{Y} + \omega^2 Y = 0$. The last equation is the well-known equation of a harmonic oscillator with two degrees of freedom. Consequently, the vibrating pattern rotates with an angular rate of $\eta\Omega$ (in the rotating reference frame $Oxyz$) in the direction of rotation of the system, if $\eta > 0$, and in the opposite direction, if $\eta < 0$.

2.1 Lagrangian of the System and Equations of Motion

The eigenfunctions and eigenvalues of the system do not change with a small inertial rotation. Hence in this section we neglect rotation. The kinetic energy of the system is

$$K=\frac{1}{2}\cdot\int_0^{2\pi}\sum_{i=1}^{N}\left[\rho_i h_i\cdot\int_{a_{i-1}}^{a_i}\left(\dot{u}_i^2+\dot{v}_i^2\right)rdr\right]d\varphi, \tag{8}$$

and the strain (or potential) energy is (see, for instance, [4])

$$P=\frac{1}{2}\cdot\int_0^{2\pi}\sum_{i=1}^{N}\left[\frac{E_i h_i}{1-\sigma_i^2}\cdot\int_{a_{i-1}}^{a_i}\left(\varepsilon_1^{(i)2}+\varepsilon_1^{(i)2}+2\sigma_i\,\varepsilon_1^{(i)}\varepsilon_2^{(i)}+\frac{1-\sigma_i}{2}\,\omega^{(i)2}\right)rdr\right]d\varphi \tag{9}$$

where σ_i is the Poisson ratio of the i^{th} disc and the strains $\varepsilon_1^{(i)}$, $\varepsilon_2^{(i)}$, $\omega^{(i)}$ are:

$$\varepsilon_1^{(i)}=\frac{\partial u_i}{\partial r},\qquad \varepsilon_1^{(i)}=\frac{1}{r}\left(u_i+\frac{\partial v_i}{\partial\varphi}\right),\qquad \omega^{(i)}=\frac{\partial v_i}{\partial r}+\frac{1}{r}\left(\frac{\partial u_i}{\partial\varphi}-u_i\right). \tag{10}$$

After substituting the expressions given in Eq. (10) into Eq. (9), the Lagrangian of the system is

$$L=K\left(\dot{u},\dot{v}\right)-P\left(u_r',v_r',u_\phi',v_\varphi',u,v\right)=L\left(\dot{u},\dot{v},u_r',v_r',u_\phi',v_\varphi',u,v\right) \tag{11}$$

Hence the equations of motion are

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial\dot{q}}\right)+\frac{\partial}{\partial r}\left(\frac{\partial L}{\partial q_r{}'}\right)+\frac{\partial}{\partial\varphi}\left(\frac{\partial L}{\partial q_\varphi{}'}\right)-\frac{\partial L}{\partial q}=0\qquad\left(q=u_i,v_i\right) \tag{12}$$

In explicit form, these equations are

$$\begin{gathered}\rho\ddot{u}-\frac{E}{1-\sigma^2}\left\{\left[u_{r,r}{}''+\frac{1}{r}u_r{}'-\frac{1}{r^2}u+\frac{1-\sigma}{2r^2}u_{\varphi,\varphi}{}''\right]+\left[\frac{1+\sigma}{2r}v_{r,\varphi}{}''-\frac{3-\sigma}{2r^2}v_\varphi{}'\right]\right\}=0;\\ \rho\ddot{v}-\frac{E}{1-\sigma^2}\left\{\left[\frac{1+\sigma}{2r}u_{r,\varphi}{}''+\frac{3-\sigma}{2r^2}u_\varphi{}'\right]+\left[\frac{1-\sigma}{2}\left(v_{r,r}{}''+\frac{1}{r}v_r{}'-\frac{1}{r^2}v\right)+\frac{1}{r^2}v_{\varphi,\varphi}{}''\right]\right\}=0\end{gathered} \tag{13}$$

for $u=u_i$ and $v=v_i$.

Using the change of variables

$$u=\frac{\partial\Phi}{\partial r}+\frac{1}{r}\frac{\partial\Psi}{\partial\varphi};\qquad v=\frac{1}{r}\frac{\partial\Phi}{\partial\varphi}-\frac{\partial\Psi}{\partial r} \tag{14}$$

separation of variables is obtained

$$\rho\ddot{\Phi}-\frac{E}{1-\sigma^2}\nabla^2\Phi=0;\qquad \rho\ddot{\Psi}-G\nabla^2\Psi=0 \tag{15}$$

where $G=E/2(1+\sigma)$ is the modulus of elasticity of the second order.

Assume that the solutions to Eq. (15) are of the form:

$$\Phi(r,\varphi,t)=\left\{A\,J_m\left[k_1(\omega)r\right]+B\,Y_m\left[k_1(\omega)r\right]\right\}\cos(m\varphi)e^{i\omega t};$$

$$\Psi(r,\varphi,t)=\left\{C\,J_m\left[k_2(\omega)r\right]+D\,Y_m\left[k_2(\omega)r\right]\right\}\sin(m\varphi)e^{i\omega t} \tag{16}$$

where $k_1(\omega)=\omega\sqrt{\rho\left(1-\sigma^2\right)/E}$, $k_2(\omega)=\omega\sqrt{\rho/G}$ are the first and second wave numbers and J_m and Y_m are Bessel and Neumann functions respectively.

If the centre of the disc O ($r=0$) is considered, then $B=D=0$. Eigenfunctions $U=U_i(r)$ and $V=V_i(r)$ in expression (3) are calculated by substituting Eq. (16) into Eq. (15). Then

$$\begin{aligned}
U(r) &= A\left\{\frac{m}{r}J_m\left[k_1(\omega)r\right]-k_1(\omega)J_{m+1}\left[k_1(\omega)r\right]\right\}+B\left\{\frac{m}{r}Y_m\left[k_1(\omega)r\right]-k_1(\omega)Y_{m+1}\left[k_1(\omega)r\right]\right\}\\
&\quad+\frac{m}{r}\left\{C\,J_m\left[k_2(\omega)r\right]+D\,Y_m\left[k_2(\omega)r\right]\right\};\\
V(r) &= \frac{m}{r}\left\{A\,J_m\left[k_1(\omega)r\right]+B\,Y_m\left[k_1(\omega)r\right]\right\}-\\
&\quad C\left\{\frac{m}{r}J_m\left[k_2(\omega)r\right]-k_2(\omega)J_{m+1}\left[k_2(\omega)r\right]\right\}+D\left\{\frac{m}{r}Y_m\left[k_2(\omega)r\right]-k_2(\omega)Y_{m+1}\left[k_2(\omega)r\right]\right\}
\end{aligned} \tag{17}$$

2.2 Boundary conditions

Let us consider an example of a hollow annular disc with inner radius $a_0=a$ and outer radius $a_1=b.$ Suppose that both radii are free. In this case the boundary conditions, which are obtained from the Lagrangian (11), are:

$$r=a:\quad \left.\frac{\partial L}{\partial u_r^{(1)\prime}}\right|_{r=a}=0;\quad \left.\frac{\partial L}{\partial v_r^{(1)\prime}}\right|_{r=a}=0;\qquad r=b:\quad \left.\frac{\partial L}{\partial u_r^{(1)\prime}}\right|_{r=b}=0;\quad \left.\frac{\partial L}{\partial v_r^{(1)\prime}}\right|_{r=b}=0 \tag{18}$$

or in explicit form:

$$\frac{E_1h_1}{1-\sigma_1^2}\left[u_r^{(1)\prime}+\frac{\sigma_1}{r}\left(u^{(1)}+mv^{(1)}\right)\right]_{r=a,b}=0;\quad G_1h_1\left[v_r^{(1)\prime}-\frac{1}{r}\left(mu^{(1)}+v^{(1)}\right)\right]_{r=a,b}=0; \tag{19}$$

If, for example, the inner boundary of the disc is fixed, then the first and second boundary conditions in Eq. (18) and Eq. (19) must be changed to

$$r=a:\qquad u\big|_{r=a}=0;\qquad v\big|_{r=a}=0 \tag{20}$$

3. EXAMPLE

Consider an aluminium disc with $E = 70\ GPa$, $\rho = 2700\ {}^{kg}/_{m^3}$, $\sigma = 0.33$, outer boundary at $r = b = 0.15\ m$ free and thickness $h = 0.01$ m (the thickness h = 0.01 m and the outer boundary of radius b = 0.15 m remain constant for all the examples).

Table 1 illustrates the dependence of Bryan's factor (Eq. (6)) on the eigenvalues of the system, for a fixed circumferential wave number $m = 2$.

Table 1. Eigenvalues (in *Hz)* and associated Bryan's factors for free inner boundaries

$m = 2$	a = 0.001	a = 0.005	a = 0.01	a = 0.05	a = 0.1	a = 0.14
f	7769.6	7719.9	7565.6	4740.5	1876.1	297.9
η	-0.8699	-0.8666	-0.8565	-0.7547	-0.7753	-0.7990

When the shell thickness becomes small ($b - a << a + b$) the effects may be described in the frames of the model of thin inextensional rings (see [5] and [6]). Using this theory and our definition of Bryan's factor we calculated the following values at $a = 0.14\ m$:

$$f = \frac{m\left(m^2 - 1\right)}{2\pi\sqrt{m^2 + 1}}\sqrt{\frac{4E\left(b - a\right)^2}{3\rho\left(a + b\right)^4}} = 298.6\ Hz, \qquad \eta = -\frac{2m}{m^2 + 1} = -0.8 \tag{21}$$

These values are very close to the values in Table 1.

Let us look at the dependence of the eigenvalues and Bryan's factor on the circumferential wave number (m). Suppose that $a = 0.05\ m$ and $b = 0.15\ m$. The data for the first five eigenvalues and for $m = 2, 3, 4, 5$ are given in the Table 2.

Table 2. Eigenvalues (in *Hz)* and associated Bryan's factors for different values of *m*

	f_1, *(Hz)* η_1	f_2 *(Hz)* η_2	f_3 *(Hz)* η_3	f_4 *(Hz)* η_4	f_5 *(Hz)* η_5
$m = 2$	4740.5 -0.7547	13657.6 0.1300	26073.5 0.0300	29133.4 0.01333	41334.7 0.3700
$m = 3$	10149.1 -0.5697	18718.6 -0.1752	27080.0 0.1289	36091.4 -0.1800	44387.6 0.3055
$m = 4$	14853.8 -0.4627	23960.2 -0.2589	30281.2 0.09547	42230.9 -0.2970	46721.2 0.2267
$m = 5$	18740.7 -0.3819	29361.0 -0.2472	35026.7 0.01308	46447.6 -0.1667	51070.3 -0.00521

Bryan's factors do not depend on thickness (h), mass density (ρ) and modulus of elasticity (E), but do depend on Poisson's ratio (σ) of the disc. This dependence is weak and is shown in Table 3 for the lowest eigenvalue of the system and $m = 2$.

Table 3. Eigenvalues (in *Hz)* and associated Bryan's factors for different values of σ

$m = 2$	σ = 0.05	σ = 0.1	σ = 0.2	σ = 0.33	σ = 0.4	σ = 0.475
f	4927.0	4892.4	4825.9	4740.5	4696.5	4650.5
η	-0.7547	-0.7547	-0.7546	-0.7547	-0.7547	-0.7547

Now suppose that the inner boundary of the disc is clamped. The results for the lowest eigenvalues and Bryan's factors for $m = 2$ are given in Table 4.

Table 4. Eigenvalues (in *Hz)* and associated Bryan's factors for clamped inner boundaries

$m = 2$	a = 0.001	a = 0.005	a = 0.01	a = 0.05	a = 0.1	a = 0.14
f	7772.709	7797.227	7872.829	10083.321	16579.695	75620.31
η	-0.87009	-0.87170	-0.87657	-0.97905	-0.36348	-0.01122

4. CONCLUSIONS

Comparing the eigenvalues and associated Bryan's factors for various free inner boundaries, it was found that Bryan's factor depends on the inner radius. Bryan's factor increases if the difference between outer and inner radii of the disc decreases (in this case the inner radius increases).

According to the second and third tables, Bryan's factor is almost independent of the Poisson ratio σ.

Comparing the eigenvalues and their associated Bryan's factor for various clamped inner boundaries, it was found that Bryan's factor decreases as the inner radius increases.

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