ICSV14
Cairns • Australia
9-12 July, 2007

14th International Congress on Sound & Vibration

INVESTIGATION OF GYROSCOPIC EFFECTS IN VIBRATING FLUID-FILLED CYLINDERS SUBJECTED TO AXIAL ROTATION

M. Shatalov[1&2], S. Joubert[2], E. Voges[2]

[1] Sensor Science and Technology (SST) of CSIR Material Science and Manufacturing (MSM),
P.O. Box 395, Pretoria 0001, CSIR, and
Department of Mathematics and Statistics
P.B.X680, Pretoria 0001 FIN-40014 Tshwane University of Technology, South Africa
e-mail: mshatlov@csir.co.za
[2] Department of Mathematics and Statistics
P.B.X680, Pretoria 0001 FIN-40014 Tshwane University of Technology, South Africa

Abstract

Vibrating patterns of distributed oscillating structures, subjected to rotation, also turn in the direction of inertial revolution but with different angular rates, which depend on the geometry of the structures, the number of modes, etc. This effect, found by G. Bryan in 1890, has numerous applications in navigational instruments such as cylindrical rotational sensors. This effect is also important in astrophysics and seismology. In the present paper we consider the main principles of the theory of gyroscopic effects in distributed structures. The model of a thick vibrating cylinder filled with a fluid and subjected to inertial rotation is analyzed. The dynamics of the cylinder is considered in terms of linear elasticity and the fluid is supposed to be ideal and inviscid, but fully involved in the rotation. It is presumed that the angular rate of inertial rotation is constant and has axial orientation. It is also assumed that the angular rate is much smaller than the lowest eigenvalue of the system and hence the centrifugal effects, proportional to square of the angular rate, are neglected. The influence of the following on Bryan's factor are investigated: the non-axisymmetric modes of the system, the eigenvalues for a fixed mode, the mass density of the fluid, the modulus of elasticity, the bulk modulus, Poisson ratio, the thickness and inner radius of the cylinder. It is shown that the difference between rotational angular rates of the system and its vibrating patterns is substantial for lower eigenvalues and circumferential wave numbers.

1. INTRODUCTION

The device used in space shuttles to measure inertial rotation in space is based on United States of America patent number 4,951,508 dated August 28, 1990. This device makes use of the so-called Bryan's effect according to which vibrating patterns in oscillating structures move in the direction of the inertial rotation, but at different angular rates.

Bryan [1] initially investigated the sound produced by a wineglass. He struck the glass to get a continuous sound and then rotated it slowly about its stem. He heard beats, which, according to him, showed that nodes revolve at an angular rate different from that of the shell. He then attempted to quantify this difference in angular rate by a quantity known as Bryan's factor. Faraday [2], Spurr [3] and Apfel [4] discussed similar observations, but in less detail.

In this work the dynamics of a fluid-filled cylinder is investigated. The influence of the mass density of the fluid, the modulus of elasticity, the bulk modulus, Poisson's ratio, and the inner radius and thickness of the cylinder is considered.

2. THE MODEL

Consider an isotropic hollow cylindrical shell consisting of a solid substance subjected to inertial rotation about its axis as shown in Figure 1. The cylinder is filled with an inviscid, compressible fluid.

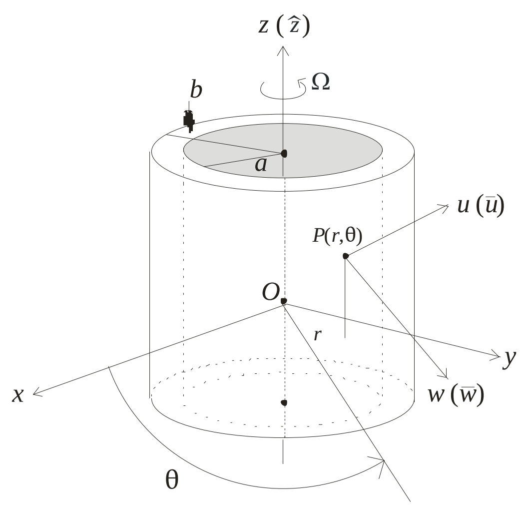

Figure 1. The fluid-filled cylinder

The angular rate of rotation of the fluid and of the cylinder coincides. In order to find the eigenvalues and eigenfunctions, the angular rotation of the system is initially disregarded. Thereafter perturbations, which are stipulated by an inertial rotation of the system about the cylinder axis, are considered. The angular rate Ω of inertial rotation is assumed to be substantially less than the first eigenvalue of the cylinder-and-fluid system so that terms proportional to Ω^2 may be disregarded, that is, centrifugal forces are neglected.

The reference frame *Oxyz* is fixed in space and $O\hat{x}\hat{y}\hat{z}$ is rotated with constant angular rate Ω relative to an inertial space (Fig. 1). The cylinder has inner radius *a* and outer radius *b*. The variables r and θ are radius and polar angle respectively. The outer boundary of the cylinder, at $r = b$, is assumed to be free of loads. Let the tangential displacement of the cylinder and fluid be $u = u(r,\theta,t)$ and $\bar{u} = \bar{u}(r,\theta,t)$ respectively. The radial displacements are $w = w(r,\theta,t)$ and $\bar{w} = \bar{w}(r,\theta,t)$ respectively, as shown in Figure 1. Assume that no axial displacement occurs. Damping is neglected since it assumed to be so small that it does not substantially affect the motion.

3. BASIC FORMULAS

Let ρ and $\bar{\rho}$ respectively represent the mass densities of the cylinder and the fluid. The Lame constants λ and μ are given by

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \text{ and } \mu = \frac{E}{2(1+\nu)}$$

where E is the modulus of elasticity and ν is Poisson's ratio. The bulk modulus of the fluid is represented by η.

Based on Hooke's law for isotropic cylinders, the Lagrangian for the system defined as the difference between kinetic energy and strain energy, and Hamilton's principle [5], the following equations and boundary conditions describe the system mathematically.

For the fluid,

$$\bar{\rho}\,\ddot{\bar{u}} - \eta\left[\frac{1}{r^2}\bar{u}_{\theta\theta} + \frac{1}{r}\bar{w}_{r\theta} + \frac{1}{r^2}\bar{w}\right] = 0 \text{ and } \bar{\rho}\,\ddot{\bar{w}} - \eta\left[\bar{w}_{rr} + \frac{1}{r}\bar{w} - \frac{1}{r^2}\bar{w} + \frac{1}{r}\bar{u}_{r\theta} - \frac{1}{r^2}\bar{u}_{\theta}\right] = 0 \qquad (1)$$

where the derivatives have the usual meaning. For the cylinder,

$$\rho\ddot{w} - (\lambda+2\mu)\left[w_{rr} + \frac{1}{r}w_r - \frac{1}{r^2}w\right] - \mu\left[\frac{1}{r}w_{\theta\theta}\right] - (\lambda+\mu)\left[\frac{1}{r}u_{r\theta}\right] + (\lambda+3\mu)\left[\frac{1}{r^2}u_{\theta}\right] = 0, \qquad (2)$$

$$\rho\ddot{u} - \mu\left[u_{rr} + \frac{1}{r}u_r - \frac{1}{r^2}u\right] - (\lambda+2\mu)\left[\frac{1}{r^2}u_{\theta\theta}\right] - (\lambda+\mu)\left[\frac{1}{r}w_{r\theta}\right] - (\lambda+3\mu)\left[\frac{1}{r^2}w_{\theta}\right] = 0\,. \qquad (3)$$

At the inner wall of the cylinder, $r = a$,

$$\bar{w} - w = 0, \quad \eta\left[\bar{w}_r + \frac{1}{r}(\bar{w} + \bar{u}_{\theta})\right] - (\lambda+2\mu)w_r - \lambda\left[\frac{1}{r}(w + u_{\theta})\right] = 0, \quad \mu\left[u_r + \frac{1}{r}(w_{\theta} - u)\right] = 0.$$

At the outer wall, $r = b$,

$$-\mu\left[u_r + \frac{1}{r}(w_{\theta} - u)\right] = 0, \quad (\lambda+2\mu)w_r + \lambda\left[\frac{1}{r}(e + u_{\theta})\right] = 0, \quad \mu\left[u_r + \frac{1}{r}(w_{\theta} - u)\right] = 0.$$

Implementation of the changes of variables

$$\bar{w} = \frac{\partial X(r,\theta,t)}{\partial r}, \quad \bar{u} = \frac{1}{r}\frac{\partial X(r,\theta,t)}{\partial \theta}, \quad w = \frac{\partial \Phi(r,\theta,t)}{\partial r} + \frac{1}{r}\frac{\partial \Psi(r,\theta,t)}{\partial \theta}, \quad u = \frac{1}{r}\frac{\partial \Phi(r,\theta,t)}{\partial \theta} - \frac{\partial \Psi(r,\theta,t)}{\partial r}$$

in Equations (1) – (3) leads, for the fluid, to

$$\ddot{X} = c^2\nabla^2 X, \quad c = \sqrt{\eta / \bar{\rho}}$$

with steady state solution

$$X(r,\theta,t) = a_1 J_m\left(\frac{\omega}{c} r\right)\cos(m\theta + \omega t) \tag{4}$$

where J_m is the first order Bessel function, ω the eigenvalues which will be determined and a_1 is a constant. For the cylinder,

$$\ddot{\Phi} - c_1^2\nabla^2\Phi = 0, \quad \ddot{\Psi} - c_2^2\nabla^2\Psi = 0$$

where $c_1 = \sqrt{(\lambda + 2\mu)/\rho}$ and $c_2 = \sqrt{\mu/\rho}$. The solutions are, with Y_m the second order Bessel function,

$$\Phi(r,\theta,t) = \left[a_2 J_m\left(\frac{\omega}{c_1} r\right) + a_3 Y_m\left(\frac{\omega}{c_1} r\right)\right]\cos(m\theta + \omega t), \tag{5}$$

$$\Psi(r,\theta,t) = \left[a_4 J_m\left(\frac{\omega}{c_2} r\right) + a_5 Y_m\left(\frac{\omega}{c_2} r\right)\right]\sin(m\theta + \omega t). \tag{6}$$

Substituting Equations (4) – (6) in Equations (1) – (3) yields

$$\overline{w} = a_1\left[J_m'\left(\frac{\omega}{c} r\right)\right]\cos(m\theta + \omega t), \quad \overline{u} = a_1\left[-\frac{m}{r} J_m\left(\frac{\omega}{c} r\right)\right]\sin(m\theta + \omega t) \tag{7}$$

$$w = a_2\left[J_m'\left(\frac{\omega}{c_1} r\right)\right] + a_3\left[Y_m'\left(\frac{\omega}{c_1} r\right)\right] + a_4\left[\frac{m}{r} J_m\left(\frac{\omega}{c_2} r\right)\right] + a_5\left[\frac{m}{r} Y_m\left(\frac{\omega}{c_2} r\right)\right]\cos(m\theta + \omega t) \tag{8}$$

$$u = a_2\left[-\frac{m}{r} J_m\left(\frac{\omega}{c_1} r\right)\right] + a_3\left[-\frac{m}{r} Y_m\left(\frac{\omega}{c_1} r\right)\right] + a_4\left[-J_m'\left(\frac{\omega}{c_2} r\right)\right] + a_5\left[-Y_m'\left(\frac{\omega}{c_2} r\right)\right]\sin(m\theta + \omega t) \tag{9}$$

where m is a positive integer representing the modal number.

To solve for ω, the eigenvalues of the system, Mathcad™ was used to solve the characteristic equation

$$\begin{vmatrix} b_{11}(\omega) & b_{12}(\omega) & b_{13}(\omega) & b_{14}(\omega) & b_{15}(\omega) \\ b_{21}(\omega) & b_{22}(\omega) & b_{23}(\omega) & b_{24}(\omega) & b_{25}(\omega) \\ 0 & b_{32}(\omega) & b_{33}(\omega) & b_{34}(\omega) & b_{35}(\omega) \\ 0 & b_{42}(\omega) & b_{43}(\omega) & b_{44}(\omega) & b_{45}(\omega) \\ 0 & b_{52}(\omega) & b_{53}(\omega) & b_{54}(\omega) & b_{55}(\omega) \end{vmatrix} = 0$$

where b_{ij}, i,j = 1, 2, 3, 4, 5 are determined by the coefficients of a_i in Equations (7) – (9). To determine the corresponding eigenfunctions, set $a_1 = 1$ and solve

$$\begin{bmatrix} b_{22}(\omega) & b_{22}(\omega) & b_{22}(\omega) & b_{22}(\omega) \\ b_{22}(\omega) & b_{22}(\omega) & b_{22}(\omega) & b_{22}(\omega) \\ b_{22}(\omega) & b_{22}(\omega) & b_{22}(\omega) & b_{22}(\omega) \\ b_{22}(\omega) & b_{22}(\omega) & b_{22}(\omega) & b_{22}(\omega) \end{bmatrix} \begin{bmatrix} a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} -b_{12}(\omega) \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

4. BRYAN'S EFFECT AND BRYAN'S FACTOR

Assume that the system is rotating slowly in an anticlockwise direction about the vertical axis with a constant angular rate of Ω, as indicated in Figure 1. Assume Ω is much smaller than the first eigenvalue so that terms in Ω^2 may be neglected.

Suppose the solutions of the equations of motion for the m^{th} mode of vibration for the fluid are

$$\overline{w}_{mn}(r,\theta,t) = \overline{W}_{mn}(r)\left[C_{mn}(t)\cos m\theta + S_{mn}(t)\sin m\theta\right],$$

$$\overline{u}_{mn}(r,\theta,t) = \overline{U}_{mn}(r)\left[-S_{mn}(t)\cos m\theta + C_{mn}(t)\sin m\theta\right]$$

where $S = S_{mn}(t)$ and $C = C_{mn}(t)$ are some functions of time, and $\overline{W}_{mn}(r)$ and $\overline{U}_{mn}(r)$ are eigenfunctions. The kinetic energy may then be written as

$$T = 2\pi\left[I_0(\dot{C}^2 + \dot{S}^2) - \Omega I_1(C\dot{S} - \dot{C}S)\right], \quad I_0 = \frac{\overline{\rho}}{2}\int_0^a (\overline{U}^2 + \overline{W}^2) r\,dr, \quad I_1 = \overline{\rho}\int_0^a \overline{W}\,\overline{U} r\,dr\,. \tag{10}$$

Substitution of the first equation in (10) and its derivatives in the Lagarnge-Euler equations of motion for the vibration yields

$$\ddot{S} - 2\Omega\eta_B\dot{C} + \omega_B^2 S = 0, \quad \ddot{C} + 2\Omega\eta_B\dot{S} + \omega_B^2 C = 0 \tag{11}$$

where $\omega_B^2 = I_2 / I_0$ and I_2 is a constant determined by the eigenfunctions in the previous section. Bryan's factor is given by $\eta_B = I_1 / I_0$ with $0 \le |\eta_B| \le 1$. Combining the equations in (11),

$$(\ddot{C} + i\ddot{S}) - 2\Omega\eta_B i(\dot{C} + i\dot{S}) + \omega_B^2(C + iS) = 0\,. \tag{12}$$

Introduce a new complex variable $R(t) = R = C + iS$, then Equation (12) becomes

$$\ddot{R} - 2i\eta_B\Omega\dot{R} + \omega_B^2 R = 0\,. \tag{13}$$

Change the variable $R \mapsto Q : R(t) = Q(t)e^{i\alpha t}$ where α is a constant to be determined. Substitution of R and its derivatives in Equation (13) yields

$$\ddot{Q} + 2i(\alpha - \Omega\eta_B)\dot{Q} + (\omega_B^2 + 2\alpha\Omega\eta_B - \alpha^2)Q = 0\,. \tag{14}$$

The term with $\dot{Q}$ in Equation (14) will be eliminated if $\alpha = \Omega\eta_B$. For this value of α, $\omega_B^2 + 2\alpha\Omega\eta_B - \alpha^2 \approx \omega_B^2$ since terms in Ω^2 are neglected. Equation (14) then simplifies to $\ddot{Q} + \omega_B^2 Q \approx 0$. This equation describes the motion of a harmonic oscillator in a reference frame rotating with an angular velocity $\hat{\Omega} = \Omega\eta_B$ relative to the reference frame $O\hat{x}\hat{y}\hat{z}$. The transformation $R(t) = Q(t)e^{i\alpha t}$ thus fixed the vibrating pattern in a reference frame rotating with an angular rate $\hat{\Omega}$ relative to $O\hat{x}\hat{y}\hat{z}$. But this reference frame is rotating with an angular rate of Ω relative to the fixed reference frame *Oxyz* in Figure 1. An observer in the fixed reference frame thus sees the vibrating pattern rotating with angular rate $\breve{\Omega} = (1+\eta_B)\Omega$.

Following the same reasoning for the whole system, it is found that Bryan's factor of the first order for the system is given by

$$\eta_B = \frac{I_1(\text{fluid}) + I_1(\text{shell})}{I_0(\text{fluid}) + I_0(\text{shell})}.$$

Since η_B depends on the value of *m*, Bryan's factor of the second order is defined as

$$B = \frac{\eta_B}{m}$$

and is calculated in the next session.

5. RESULTS

The eigenvalues for non-axisymmetric nodes are considered in the audible frequency range, that is, 20 Hz to 30 kHz. Initially the following geometrical and physical parameters were used: the radii are $a = 20$ mm and $b = 25$ mm, the bulk modulus is $\eta = 2.25\times10^9$ N/m^2, the mass densities of the fluid and the liquid are $\bar{\rho} = 10^3$ kg/m^3 and $\rho = 2.7\times10^3$ kg/m^3 respectively, the Lame constants are $\lambda = 1.761\times10^{10}$ N/m^2 and $\mu = 3.419\times10^{10}$ kg/m^2. Young's modulus is $E = 8\times10^{10}$ and Poisson's ratio is $\nu = 0.17$. Hence, $c = 1.5\times10^3$ m/s, $c_1 = 5.643\times10^3$ m/s and $c_2 = 3.558\times10^3$ m/s.

In Table 1 it can be seen that *B* gets closer to 0 for increasing values of *m*. In the rest of the investigation *m* = 4 was used. Table 2 shows the influence of the eigenvalues for a fixed mode, *m* = 4, on *B*. For the results in Table 3 the outer radius was fixed and the inner radius was changed, that is, the thickness of the shell changed. To generate the data in Table 4, the thickness if the shell remained the same, while the inner radius of the shell increased. Tables 5, 6, 7 and 8 reflects the influence of the Poisson ratio, the bulk modulus, Young's modulus and the mass density of the fluid respectively.

Table 1. Bryan's factor, *B*, for the first eigenvalue of each value of *m*

m	2	3	4	5	6	8
B	-0.432	-0.236	-0.151	-0.108	-0.086	-0.068

Table 2. *B* for changing eigenvalues (*f*)

f (Hz)	2.845×10^{3}	6.951×10^{3}	1.123×10^{4}	1.510×10^{4}	1.596×10^{4}	1.913×10^{4}
B	-0.151	-0.120	-0.035	-0.020	0.096	-0.004

Table 3. The influence of the thickness (mm) of the shell on *B*

Thickness	1	5	10	15
B	-0.207	-0.151	-0.132	-0.110

Table 4. The influence of the inner radius (mm) of the shell on *B*

a	100	200	300	400
B	-0.137	-0.151	-0.162	-0.170

Table 5. The influence of the Poisson ratio on *B*

ν	0.05	0.10	0.17	0.2	0.4
B	-0.151	-0.151	-0.151	-0.150	-0.148

Table 6. The influence of the bulk modulus on *B*

$\eta\,(\times10^{9})$	1.00	2.00	2.25	2.50	3.00
B	-0.163	-0.152	-0.151	-0.150	-0.148

Table 7. The influence of Young's modulus on *B*

$E\,(\times10^{10})$	6	7	8	9	10
B	-0.148	-0.149	-0.151	-0.152	-0.153

Table 8. The influence of the fluid's mass density on *B*

$\overline{\rho}\,(\times10^{3})$	0.8	1.0	1.2	1.5	2.0
B	-0.143	-0.151	-0.157	-0.166	-0.178

6. CONCLUSIONS

From Tables 1 and 2 it is seen that both the mode numbers *m* and the eigenvalues contribute towards Bryan's factor. The influence of both is substantial for lower values. Note the positive value of the fifth eigenvalue in Table 2. This implies that the vibrating pattern will precede the rotating body for this eigenvalue, while the pattern will lag for the other eigenvalues.

The thicker the shell, the smaller is the Bryan's factor, as seen in Table 3. This implies that Bryan's effect is less for thicker shells. On the other hand, the bigger the inner radius of the shell, the bigger Bryan's effect as is seen in Table 4.

Bigger values of both the Poisson ratio (Table 5) and the bulk modulus (Table 6) lead to Bryan's factors closer to 0. Increased values of Young's modulus (Table 7) and the mass density of the fluid (Table 8) lead to more pronounced Bryan's effects.

REFERENCES

[1] G.H. Bryan, "On the beats in the vibrations of a revolving cylinder or bell", *Proceedings of the Cambridge Philosophical Society for Mathematical and Physical Sciences* **7**,101-111 (1890).

[2] M. Faraday, "On a peculiar class of acoustical figures and on certain forms assumed by groups of particles upon vibrating elastic surfaces", *Philosophical Transactions of the Royal Society of London* **XVII**, 299-340 (1831).

[3] R.T. Spurr, "The ringing of wineglasses", *Wear* **4(2)**, 150-153 (1961).

[4] R.E. Apfel, "Whispering waves in a wineglass", *American Journal of Physics Teachers* **53(11)**, 1070-1073 (1984).

[5] J.L. Davis, *Mathematics of wave propagation*. Princeton University Press (2000).