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**VIBRATORY GYROSCOPES: IDENTIFICATION OF
MATHEMATICAL MODEL FROM TEST DATA**

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Abstract

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Simple mathematical model of vibratory gyroscopes imperfections is formulated, which includes anisotropic damping and variation of mass-stiffness parameters and their harmonics. The method of identification of parameters of the mathematical model from the experimental data is based on transformation of the system of linear differential equations of the model into an overdetermined system of linear algebraic equations with subsequent matching of the system parameters by means of the least squares method. Example of practical calculations of parameters of a vibratory gyroscope is considered and it is shown by direct solution of equations of motion that the method gives a good results.

Introduction

Creation of mathematical model of gyroscope's imperfections is a crucial stage of development of its theory. A complete theory of instrument's imperfection helps to substantially improve the gyro accuracy by means of corrections of output data, which could be made on the basis of this theory. This is especially important for gyroscopes of inertial class that need a detailed and well developed theory of operations and imperfections. Another problem consists in correct identification of the model's parameters from a series of laboratory tests. Last but not least is the development of methods for fast and simple on-board identification of parameters and their changes without tedious and time consuming tests. It could be achieved by means of formulation of simple but accurate enough theory of gyro imperfections with corresponding algorithms of their on-board identification. The proposed paper represents a simple but detailed mathematical model of vibratory gyroscopes imperfections and methods of its parametric identification. This model includes the main factors that influence the vibratory gyro accuracy such as: Q-factor and deviation of its harmonics (anisotropic damping) and variation of mass-stiffness parameters and their harmonics (anisotropy of mass-stiffness parameters). Deviation of the Q-factor' harmonics causes a substantial drift of the vibrating pattern in the direction of its minimum damping axis. Hence, it is also necessary to identify location of the minimum-maximum damping axes of the gyroscope, which could change their orientation in time. Mass-stiffness harmonic imperfections stipulate growth of quadrature signals and long-periodic beats of the vibrating pattern with regards to the main mass-stiffness axes that could also change their orientation in time and hence, must be properly identified. It is shown in this paper that the abovementioned effects could be simply classified and corresponding coefficients of the mathematical model could be identified. These methods are formulated as an algorithm, main idea of which consists in time integration of the in-phase and quadrature signals from both information channels of the vibratory gyroscope at different time instants; this process generates an overdetermined system of linear algebraic equations with regards to unknown coefficients of the mathematical model of gyro imperfections. Then the system is solved by means of the least squares method and the coefficients of the model are determined. The corresponding physical

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parameters of the gyroscope, such as damping (Q-factor), frequency splitting, angles of maximum-minimum damping and mass-stiffness axes orientation, cross-coupling of input angular rate channels are estimated from the coefficients of the model in the explicit form. The example is considered and the conclusion is formulated that the method could be realized as an on-board correction algorithm.

Mathematical Model

Let us consider the following model of a vibratory gyroscope:

$$\begin{aligned} \ddot{x} + 2\delta(1 + \Delta_1 \cos 4\varphi_1)\dot{x} + 2\delta\Delta_1 \sin 4\varphi_1 \dot{y} + \omega^2(1 + \Delta_2 \cos 4\varphi_2)x + \omega^2\Delta_2 \sin 4\varphi_2 y &= 0 \\ \ddot{y} + 2\delta(1 - \Delta_1 \cos 4\varphi_1)\dot{y} + 2\delta\Delta_1 \sin 4\varphi_1 \dot{x} + \omega^2(1 - \Delta_2 \cos 4\varphi_2)y + \omega^2\Delta_2 \sin 4\varphi_2 x &= 0 \end{aligned} \quad (1)$$

where $x = x(t)$, $y = y(t)$ – output signals from X - and Y -channels, δ – damping factor, Δ_1 – coefficient of damping anisotropy, φ_1 – angle of location of the maximum damping axis with regards to the X -channel axis, ω – natural circular frequency of an operational mode (in our case the circumferential wave number $m = 2$ is considered), Δ_2 – coefficient of mass-density anisotropy, φ_2 – angle of location of the maximum frequency axis with regards to the X -channel axis. It is supposed that terms, proportional to δ and $\omega^2\Delta_2$ are supposed to be small. We assume that demodulation of $x(t)$ and $y(t)$ output signals are realized with frequency $\nu \approx \omega$ so that difference $\nu - \omega$ is also small. Writing all small terms in the right hand sides of equations (1) we obtain the following system:

$$\begin{aligned} \ddot{x} + \nu^2 x &= -2\delta(1 + \Delta_1 \cos 4\varphi_1)\dot{x} - 2\delta\Delta_1 \sin 4\varphi_1 \dot{y} + (\nu^2 - \omega^2 + \omega^2\Delta_2 \cos 4\varphi_2)x - \omega^2\Delta_2 \sin 4\varphi_2 y \\ \ddot{y} + \nu^2 y &= -2\delta(1 - \Delta_1 \cos 4\varphi_1)\dot{y} - 2\delta\Delta_1 \sin 4\varphi_1 \dot{x} + (\nu^2 - \omega^2 + \omega^2\Delta_2 \cos 4\varphi_2)y - \omega^2\Delta_2 \sin 4\varphi_2 x. \end{aligned} \quad (2)$$

We search for solution of (2) in the form:

$$\begin{aligned} x &= a \cos \psi + b \sin \psi; & y &= c \cos \psi + d \sin \psi; \\ \dot{x} &= \nu(-a \sin \psi + b \cos \psi); & \dot{y} &= \nu(-c \sin \psi + d \cos \psi), \end{aligned} \quad (3)$$

where a, b, c, d – “slowly” changing parameters of time and phase $\psi = \nu t + \psi_0$ ($\dot{\psi} = \nu$). Formulae (3) could be considered as change of variables $(x, \dot{x}, y, \dot{y}) \rightarrow (a, b, c, d)$ and in new parameters system (2) is rewritten as follows:

$$\begin{aligned} \dot{a} &= \alpha_{11}a + \alpha_{12}b + \alpha_{13}c + \alpha_{14}d + f_a(\sin 2\psi, \cos 2\psi), \\ \dot{b} &= -\alpha_{12}a + \alpha_{11}b - \alpha_{14}c + \alpha_{13}d + f_b(\sin 2\psi, \cos 2\psi), \\ \dot{c} &= \alpha_{13}a + \alpha_{14}b + \alpha_{33}c + \alpha_{34}d + f_c(\sin 2\psi, \cos 2\psi), \\ \dot{d} &= -\alpha_{14}a + \alpha_{13}b - \alpha_{34}c + \alpha_{33}d + f_d(\sin 2\psi, \cos 2\psi), \end{aligned} \quad (4)$$

where

$$\alpha_{11} = -\delta \frac{\omega}{\nu} (1 + \Delta_1 \cos 4\varphi_1), \quad \alpha_{12} = \frac{\omega^2}{2\nu} (\Delta_0 + \Delta_2 \cos 4\varphi_2), \quad \Delta_0 = \frac{\omega^2 - \nu^2}{\omega^2}, \quad (5)$$

$$\alpha_{13} = -\delta \frac{\omega}{\nu} \Delta_1 \sin 4\varphi_1, \quad \alpha_{14} = \frac{\omega^2}{2\nu} \Delta_2 \sin 4\varphi_2, \quad \alpha_{33} = -\delta \frac{\omega}{\nu} (1 - \Delta_1 \cos 4\varphi_1), \quad \alpha_{34} = \frac{\omega^2}{2\nu} (\Delta_0 - \Delta_2 \cos 4\varphi_2).$$

Expressions $f_{a,b,c,d}(\sin 2\psi, \cos 2\psi)$ – are changing fast with the double phase ψ and following to the method of averaging could be neglected in the first approximation. Residual slowly changing terms determine an evolutionary motion of the system. The averaged system:

$$\begin{aligned} \dot{a} &= \alpha_{11}a + \alpha_{12}b + \alpha_{13}c + \alpha_{14}d, & \dot{b} &= -\alpha_{12}a + \alpha_{11}b - \alpha_{14}c + \alpha_{13}d, \\ \dot{c} &= \alpha_{13}a + \alpha_{14}b + \alpha_{33}c + \alpha_{34}d, & \dot{d} &= -\alpha_{14}a + \alpha_{13}b - \alpha_{34}c + \alpha_{33}d \end{aligned} \quad (6)$$

is linear with regards to six unknown parameters $\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{33}, \alpha_{34}$.

Identification of Parameters

To define the unknown parameters from the output data of the transient regime first integrate all four equations (6) with regards to time t to obtain:

$$\begin{aligned} \alpha_{11}I_a(t) + \alpha_{12}I_b(t) + \alpha_{13}I_c(t) + \alpha_{14}I_d(t) &= \Delta_a(t) \\ -\alpha_{12}I_a(t) + \alpha_{11}I_b(t) - \alpha_{14}I_c(t) + \alpha_{13}I_d(t) &= \Delta_b(t) \\ \alpha_{13}I_a(t) + \alpha_{14}I_b(t) + \alpha_{33}I_c(t) + \alpha_{34}I_d(t) &= \Delta_c(t) \\ -\alpha_{14}I_a(t) + \alpha_{13}I_b(t) - \alpha_{34}I_c(t) + \alpha_{33}I_d(t) &= \Delta_d(t) \end{aligned} \quad (7)$$

$$\begin{aligned} I_a(t) &= \int_0^t a(\tau) d\tau; \quad I_b(t) = \int_0^t b(\tau) d\tau; \quad I_c(t) = \int_0^t c(\tau) d\tau; \quad I_d(t) = \int_0^t d(\tau) d\tau; \\ \Delta_a(t) &= a(t) - a(0); \quad \Delta_b(t) = b(t) - b(0); \quad \Delta_c(t) = c(t) - c(0); \quad \Delta_d(t) = d(t) - d(0) \end{aligned} \quad (8)$$

Integration helps us to filter out possible random errors of measurements and accumulate the information about the global evolution of the system. For the purposes of numerical integration one can use any quadrature formulae, for example trapezoidal or Simpson's rules.

Expressions (7) give us an overdetermined system of linear algebraic equations for measurement instants $t = t_1 > 0$, $t = t_2 > t_1$, ..., $t = t_N > t_{N-1}$, where $t \in [0, t_N]$ – interval of measurements and $N+1$ – number of measurements, including initial values at $t = t_0 = 0$.

Unknown parameters could be determined from this system, using the least squares method. For application of this method we form the goal function, subjected to minimization:

$$\begin{aligned} F &= F(\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{33}, \alpha_{34}) = \\ &= \frac{1}{2} \sum_{i=1}^N \left\{ \left(\alpha_{11}I_a^{(i)} + \alpha_{12}I_b^{(i)} + \alpha_{13}I_c^{(i)} + \alpha_{14}I_d^{(i)} - \Delta_a^{(i)} \right)^2 + \left(-\alpha_{12}I_a^{(i)} + \alpha_{11}I_b^{(i)} - \alpha_{14}I_c^{(i)} + \alpha_{13}I_d^{(i)} - \Delta_b^{(i)} \right)^2 + \right. \\ &\quad \left. + \left(\alpha_{13}I_a^{(i)} + \alpha_{14}I_b^{(i)} + \alpha_{33}I_c^{(i)} + \alpha_{34}I_d^{(i)} - \Delta_c^{(i)} \right)^2 + \left(-\alpha_{14}I_a^{(i)} + \alpha_{13}I_b^{(i)} - \alpha_{34}I_c^{(i)} + \alpha_{33}I_d^{(i)} - \Delta_d^{(i)} \right)^2 \right\} \end{aligned} \quad (9)$$

where:

$$I_a^{(i)} = I_a(t_i), \quad I_b^{(i)} = I_b(t_i), \quad I_c^{(i)} = I_c(t_i), \quad I_d^{(i)} = I_d(t_i), \quad \Delta_a^{(i)} = \Delta_a(t_i), \quad \Delta_b^{(i)} = \Delta_b(t_i), \quad \Delta_c^{(i)} = \Delta_c(t_i), \quad \Delta_d^{(i)} = \Delta_d(t_i).$$

Minimum of function $F = F(\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{33}, \alpha_{34})$ is achieved at parameters $\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{33}, \alpha_{34}$, satisfying the system of equations:

$$\frac{\partial F}{\partial \alpha_{11}} = \frac{\partial F}{\partial \alpha_{12}} = \frac{\partial F}{\partial \alpha_{13}} = \frac{\partial F}{\partial \alpha_{14}} = \frac{\partial F}{\partial \alpha_{33}} = \frac{\partial F}{\partial \alpha_{34}} = 0 \quad (10)$$

Explicit solution of system (10) is given by the formula:

$$\begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{13} \\ \alpha_{14} \\ \alpha_{33} \\ \alpha_{34} \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} & M_{16} \\ & M_{22} & M_{23} & M_{24} & M_{25} & M_{26} \\ & & M_{33} & M_{34} & M_{35} & M_{36} \\ & & & (Symm) & M_{44} & M_{45} & M_{46} \\ & & & & & M_{55} & M_{56} \\ & & & & & & M_{66} \end{bmatrix}^{-1} \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{bmatrix} \quad (11)$$

where

$$\begin{aligned}
M_{11} &= \sum_{i=1}^N (I_a^{(i)2} + I_b^{(i)2}); & M_{12} &= 0; & M_{13} &= \sum_{i=1}^N (I_a^{(i)} I_c^{(i)} + I_b^{(i)} I_d^{(i)}); & M_{14} &= \sum_{i=1}^N (I_a^{(i)} I_d^{(i)} - I_b^{(i)} I_c^{(i)}); \\
M_{15} &= M_{16} = 0; & M_{22} &= M_{11}; & M_{23} &= -M_{14}; & M_{24} &= M_{13}; \\
M_{25} &= M_{26} = 0; & M_{33} &= \sum_{i=1}^N (I_a^{(i)2} + I_b^{(i)2} + I_c^{(i)2} + I_d^{(i)2}); & M_{34} &= 0; & M_{35} &= M_{13}; \\
M_{36} &= M_{14}; & M_{44} &= M_{33}; & M_{45} &= -M_{14}; & M_{46} &= M_{13}; \\
M_{55} &= \sum_{i=1}^N (I_c^{(i)2} + I_d^{(i)2}); & M_{56} &= 0; & M_{66} &= M_{55}; \\
R_1 &= \sum_{i=1}^N (I_a^{(i)} \Delta_a^{(i)} + I_b^{(i)} \Delta_b^{(i)}); & R_2 &= \sum_{i=1}^N (I_b^{(i)} \Delta_a^{(i)} - I_a^{(i)} \Delta_b^{(i)}); & R_3 &= \sum_{i=1}^N (I_a^{(i)} \Delta_c^{(i)} + I_b^{(i)} \Delta_d^{(i)} + I_c^{(i)} \Delta_a^{(i)} + I_d^{(i)} \Delta_b^{(i)}); \\
R_4 &= \sum_{i=1}^N (I_b^{(i)} \Delta_c^{(i)} - I_c^{(i)} \Delta_b^{(i)} + I_d^{(i)} \Delta_a^{(i)} - I_a^{(i)} \Delta_d^{(i)}); & R_5 &= \sum_{i=1}^N (I_c^{(i)} \Delta_c^{(i)} + I_d^{(i)} \Delta_d^{(i)}); & R_6 &= \sum_{i=1}^N (I_d^{(i)} \Delta_c^{(i)} - I_c^{(i)} \Delta_d^{(i)}).
\end{aligned} \tag{12}$$

Now it is possible to calculate unknown values: frequency of the resonator ($f = \omega/2\pi$), frequency splitting ($\Delta f = 1/2\pi \cdot \omega \Delta_2$), orientation of the mass-stiffness defect (angle φ_2), damping factor (δ), Q-factor ($Q = \pi f / \delta$), damping factor defect ($\Delta \delta = \delta \Delta_1$), Q-factor defect ($\Delta Q = \pi f \Delta_1 / \delta$) and orientation of the damping factor defect (angle φ_1). These parameters could be calculated from expressions (5) by the formulae:

$$\begin{aligned}
f &= \frac{1}{2\pi} \sqrt{\nu(\nu + \alpha_{12} + \alpha_{34})}; & \Delta f &= \frac{1}{2\pi} \sqrt{\frac{\nu(\alpha_{12} - \alpha_{34})^2 + 4\alpha_{14}^2}{\nu + \alpha_{12} + \alpha_{34}}}; \\
\cos 4\varphi_2 &= \frac{\alpha_{12} - \alpha_{34}}{\sqrt{(\alpha_{12} - \alpha_{34})^2 + 4\alpha_{14}^2}}; & \sin 4\varphi_2 &= \frac{2\alpha_{14}}{\sqrt{(\alpha_{12} - \alpha_{34})^2 + 4\alpha_{14}^2}}; \\
\delta &= -\frac{\nu(\alpha_{11} + \alpha_{33})}{2\sqrt{\nu(\nu + \alpha_{12} + \alpha_{34})}}; & Q &= -\frac{\nu + \alpha_{12} + \alpha_{34}}{\alpha_{11} + \alpha_{33}}; \\
\Delta \delta &= \frac{\nu \sqrt{(\alpha_{11} - \alpha_{33})^2 + 4\alpha_{13}^2}}{2\sqrt{\nu(\nu + \alpha_{12} + \alpha_{34})}}; & \Delta Q &= \frac{(\nu + \alpha_{12} + \alpha_{34}) \sqrt{(\alpha_{11} - \alpha_{33})^2 + 4\alpha_{13}^2}}{(\alpha_{11} + \alpha_{33})^2}; \\
\cos 4\varphi_1 &= \frac{\alpha_{33} - \alpha_{11}}{\sqrt{(\alpha_{11} - \alpha_{33})^2 + 4\alpha_{13}^2}}; & \sin 4\varphi_1 &= \frac{-2\alpha_{13}}{\sqrt{(\alpha_{11} - \alpha_{33})^2 + 4\alpha_{13}^2}}.
\end{aligned} \tag{13}$$

Example

Transient regime of decaying oscillations of a vibratory gyroscope is shown in Fig.1,2. In these figures the demodulated in-phase and quadrature components of two orthogonal output channels are shown. The demodulation is performed with circular frequency $\nu = 2\pi \cdot 6143.15 \text{ s}^{-1}$ at time interval $t \in [0-180] \text{ s}$ with rate of twenty measurements per second (time increment $\Delta t = 0.05 \text{ s}$). Using the abovementioned algorithm (11) – (13) the following parameters of the vibratory gyroscope have been calculated:

- Natural frequency of the mode: $f \approx 6143.140 \text{ Hz}$.
- Frequency splitting: $\Delta f \approx 0.018 \text{ Hz}$.

- Damping factor: $\delta \approx 0.015 \text{ s}^{-1}$.
- Q – factor: $Q \approx 1.293 \cdot 10^6$.
- Damping anisotropy: $\Delta\delta \approx 0.0022$.
- Q - factor anisotropy: $\Delta Q \approx 0.1925 \cdot 10^6$.
- Angle of maximum damping axis (from X-channel axis): $\varphi_1 \approx 21.39^\circ$.
- Angle of maximum frequency axis (from X-channel axis): $\varphi_2 \approx -37.06^\circ$.

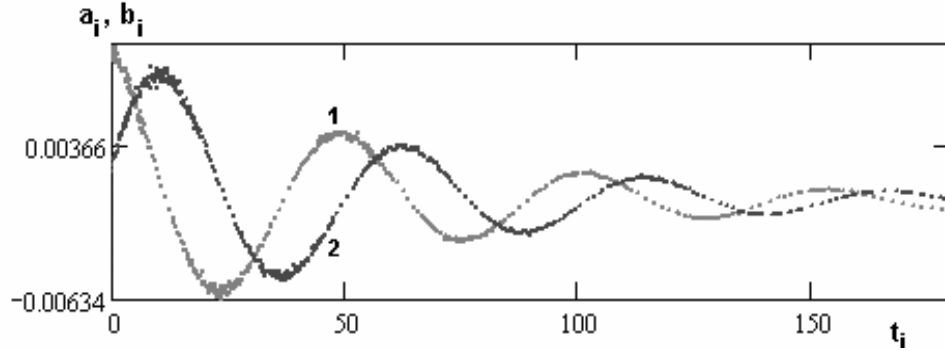


Fig. 1. Test data from channel X (1 -in-phase component, a_i ; 2 -quadrature component, b_i)

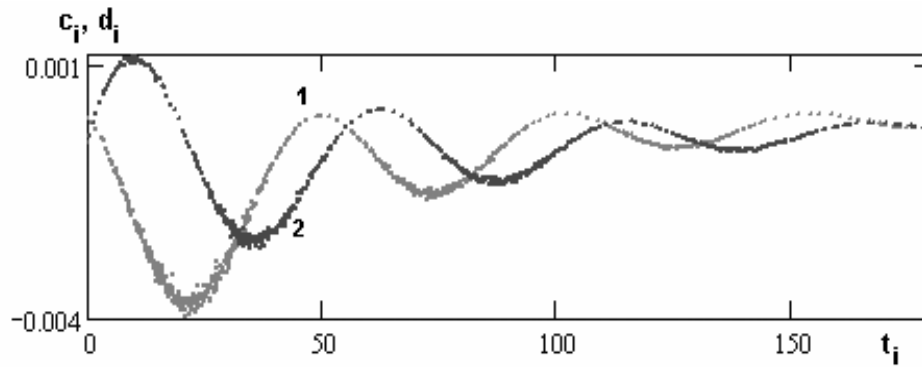


Fig. 2. Test data from channel Y (1 -in-phase component, c_i ; 2 -quadrature component, d_i)

To check the accuracy of approximations the numerical solutions of the system (6) have been calculated by an adaptive Runge-Kutta method with the same initial conditions as from the test data. Results of comparison of the test data and results of numerical integration of equations (6) are shown in Fig.3-6.

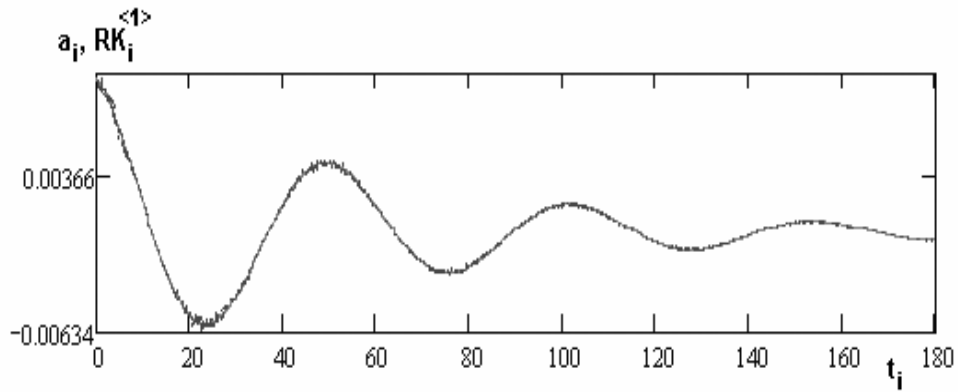


Fig. 3. Runge-Kutta and test data of in-phase X-channel components

(---- - Runge-Kutta integration; - - - - test data)

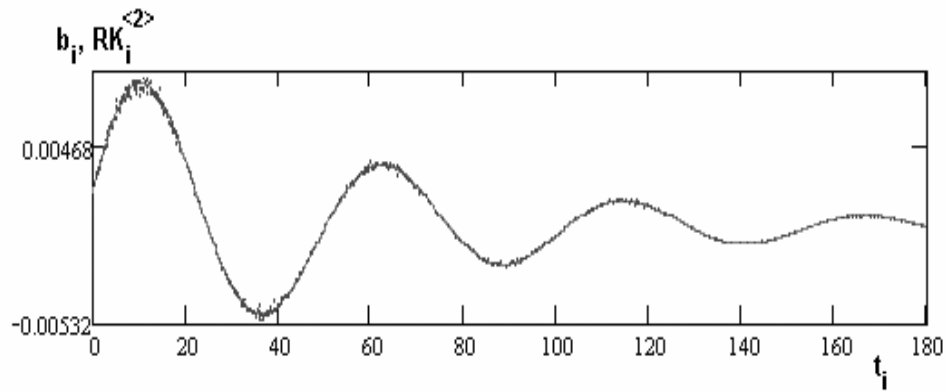


Fig. 4. Runge-Kutta and test data of quadrature X-channel components
(---- - Runge-Kutta integration; - - - - test data)

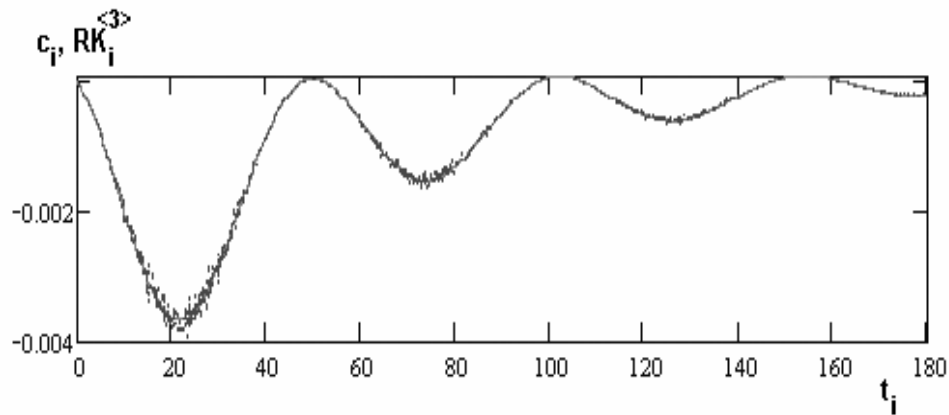


Fig. 5. Runge-Kutta and test data of in-phase Y-channel components
(---- - Runge-Kutta integration; - - - - test data)

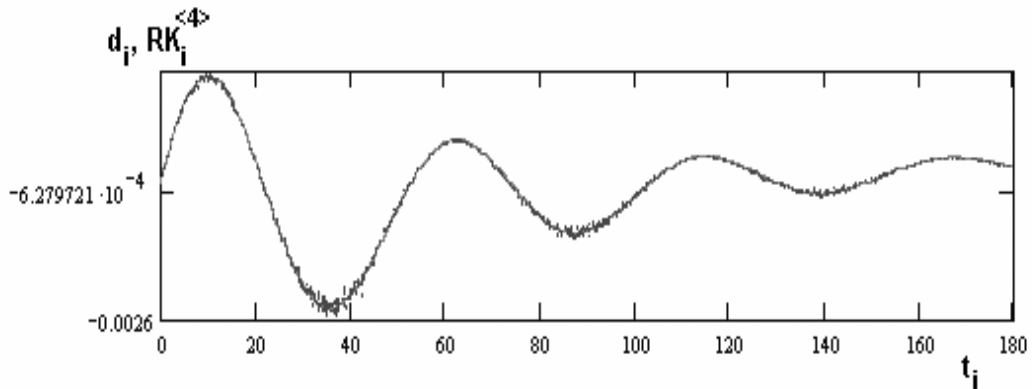


Fig. 6. Runge-Kutta and test data of quadrature Y-channel components
(---- - Runge-Kutta integration; - - - - test data)

Conclusions

The proposed algorithm of vibratory gyroscope parametric identification is formulated. It is shown that the predicted parameters give satisfactory accuracy of the experimental data interpolation. The developed methods could be used for fast characterization of the system on the stage of performing of balancing operations and periodical tests of the system.