Fifth South African Conference on Computational and Applied Mechanics
SACAM06
Cape Town, 16-18 January 2006

EXCITATION OF WAVES IN ELASTIC WAVEGUIDES BY PIEZOELECTRIC PATCH ACTUATORS

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Key words: Piezoelectric patch actuator, elastic waveguide, finite element method

Abstract

Piezoelectric patch actuators are employed for monitoring the health of structures and for performing active control of vibration. When the structure is long, slender and has constant cross-section, such as a pipe, rail or rod, it can be considered to be an infinite waveguide. The excitation of waves in waveguides may be analysed in the time domain using conventional finite element methods. This analysis is computationally very demanding as the model must be a number of wavelengths long to avoid the influence of reflections from the ends of the model. In addition, separating the contributions of the individual modes of propagation can be difficult. This paper presents a highly efficient analysis method for computing the waves excited as a function of frequency for waveguides excited by piezoelectric patch actuators. The waveguide is modelled using specially developed waveguide finite elements. These elements are formulated using a complex exponential to describe the wave propagation along the structure and finite element interpolation over the area of the element. Therefore only a two-dimensional finite element mesh covering the cross-section of the waveguide is required. The response of the waveguide to harmonic forces at the nodes which are in contact with the piezoelectric patch can be computed to produce a stiffness matrix (complex and frequency dependent) representing the waveguide. This matrix is included in a model of the piezoelectric patch developed using conventional three-dimensional piezoelectric finite elements. The response of the piezoelectric patch actuator (attached to the waveguide) to electrical excitation is computed and the displacements at the interface degrees of freedom are used to compute the forces applied to the waveguide and hence the waves excited in the waveguide. The method is applied to a patch actuator on an infinite beam of rectangular cross-section, which has been studied analytically, to verify the method. A model of a rail, excited by piezoelectric patches, is used to demonstrate the ability of the method to analyse the excitation of high frequency waves in waveguides with complex cross-sections. This modelling technique would be useful during the design of piezoelectric patches and arrays of patches for excitation of a specific mode of wave propagation.

1. INTRODUCTION

Guided wave ultrasonics is an emerging technology that offers various advantages over conventional methods [1]. Guided waves, with frequencies typically up to 200 kHz, can be used to inspect or monitor large areas of structures in structural health monitoring systems. The guided waves can be transmitted and received by piezoelectric patches bonded to or embedded in the structure. The actuation of finite structures, by the piezoelectric patch actuators, has been analysed by various authors using models of varying complexity. Excitation of waves in infinite rectangular beams was analysed by Gibbs and Fuller [2, 3] and Giurgiutiu [4] analysed Lamb wave generation in infinite plates. When the structure is long, slender and has constant cross-section, such as a pipe, rail or rod, it can be considered to be an infinite waveguide. The excitation of waves in waveguides of arbitrary cross-section may be analysed in the time domain using conventional finite element methods. This analysis is computationally very demanding as the model must be a number of wavelengths long to avoid the influence of reflections from the ends of the model. In addition, separating the contributions of the individual modes of propagation can be difficult. This paper presents a highly efficient analysis method for computing the waves excited as a function of frequency for waveguides excited by piezoelectric patch actuators.

Waves, which propagate in cylindrical waveguides, may be determined analytically [5]. The low-frequency longitudinal and bending waves in rectangular waveguides may be analysed by using Euler-Bernoulli beam theory. Numerical solutions are required at higher frequencies and for waveguides of more complex cross-section. Efficient finite element methods, for the analysis of wave in waveguides, have been implemented by Gavrić [6], Hayashi [7], and Damljanović and Weaver [8]. In these methods a complex exponential function was used to represent the displacement variation along the waveguide and finite element discretization was applied over the cross-section. This results in a two-dimensional mesh, which is capable of modeling the wave propagation along the waveguide. Wave propagation in waveguides of arbitrary cross-section, at high frequencies, can therefore be analyzed. The formulation of these elements is briefly described in section 2. Although not used in this paper, these elements can also be formulated to include piezoelectric waveguides [9].

Damljanović and Weaver [10] developed the solution for response of a waveguide to a harmonic force. In section 3 this solution is used to derive the receptance of a waveguide. This receptance can be included in a conventional finite element model of a piezoelectric patch to compute the frequency response of the patch attached to the waveguide. In addition the response of each wave, or mode of propagation, can be determined. The method is applied to piezoelectric patches on a rectangular beam and on a rail. The results, of these analyses, are presented in section 4.

2. FINITE ELEMENT FORMULATION

This section provides the essential details of the formulations of the finite elements used in this paper.

2.1 Waveguide Finite Element Formulation

The formulation used follows that presented by Gavrić [6]. The formulations of Hayashi [7] and Damljanović and Weaver [8] are similar to this formulation. Hayashi's formulation results in complex equations of motion while Damljanović and Weaver also derived complex equations of motion and then applied a transformation to obtain equations of motion comparable to those of Gavrić. The displacement field in an elastic waveguide, extending in the z direction, is described by a complex exponential along the waveguide and a finite element approximation over the cross-section. The displacement fields (u, v, w) can be written as:

$$
\begin{aligned}
u(x,y,z,t) &= u(x,y)\cdot e^{-j(\kappa z-\omega t)}\\
v(x,y,z,t) &= v(x,y)\cdot e^{-j(\kappa z-\omega t)}\\
w(x,y,z,t) &= j\cdot w(x,y)\cdot e^{-j(\kappa z-\omega t)}
\end{aligned}
\tag{1}
$$

where, z is the coordinate in the direction along the waveguide, κ the wavenumber and ω the frequency.

The strain and strain energy of the waveguide can be separated into terms that are independent, linearly dependent or quadratically dependent on the wavenumber (* indicates complex conjugate transpose).

$$
\varepsilon = \varepsilon_0 + \kappa\varepsilon_1
$$

$$
\varepsilon = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \kappa w \\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial z} \\ -j\kappa v + j\dfrac{\partial w}{\partial y} \\ -j\kappa u + j\dfrac{\partial w}{\partial x} \end{Bmatrix};\quad
\varepsilon_0 = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ 0 \\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial z} \\ j\dfrac{\partial w}{\partial y} \\ j\dfrac{\partial w}{\partial x} \end{Bmatrix};\quad
\varepsilon_1 = \begin{Bmatrix} 0 \\ 0 \\ w \\ 0 \\ -jv \\ -ju \end{Bmatrix};
\tag{2}
$$

$$
\begin{aligned}
k_0 &= \varepsilon_0 c \varepsilon_0^{\ *}\\
k_1 &= \varepsilon_0 c \varepsilon_1^{\ *} + \varepsilon_1 c \varepsilon_0^{\ *}\\
k_2 &= \varepsilon_1 c \varepsilon_1^{\ *}
\end{aligned}
\tag{3}
$$

Applying conventional finite element discretisation to these terms yields the elemental mass and stiffness matrices.

$$
\begin{aligned}
k_0 &= \int B_0 c B_0^{\ *} \det J \; d\xi d\eta\\
k_1 &= \int \left[B_0 c B_1^{\ *} + B_1 c B_0^{\ *}\right] \det J \; d\xi d\eta\\
k_2 &= \int B_1 c B_1^{\ *} \det J \; d\xi d\eta\\
m &= \int N \rho N^{*} \det J \; d\xi d\eta
\end{aligned}
\tag{4}
$$

Assembly of the element matrices produces the system equations of motion for the waveguide.

$$
M\,\ddot{u} + \left[\kappa^2 \cdot K_2 + \kappa \cdot K_1 + K_0\right] u = f \tag{5}
$$

2.2 Piezoelectric Finite Element Formulation

Finite elements for piezoelectric materials were developed by Allik and Hughes [11] and the notation adopted here is similar to theirs. The set of constitutive equations used for the piezoelectric material are,

$$
\begin{aligned}
T &= c^E S - e^t E\\
D &= eS + \varepsilon^S E
\end{aligned}
\tag{6}
$$

where, T and S are stress and strain tensors, E is the electrical field vector, D is the electric displacement vector and the t superscript indicates matrix transpose.

After application of finite element discretisation, elemental matrices are obtained. These matrices are assembled to form global matrices describing the equations of motion having the following form [11]:

$$\begin{aligned} M\ddot{u} + K_{uu}u + K_{u\Phi}\Phi &= F \\ K_{\Phi u}u + K_{\Phi\Phi}\Phi &= Q \end{aligned} \tag{7}$$

where, M and K_{uu} are mechanical mass and stiffness matrices, $K_{u\Phi} = K_{\Phi u}{}^{t}$ is the piezoelectric coupling matrix, $K_{\Phi\Phi}$ is the capacitance matrix, u and Φ are nodal displacements and electrical potentials and F and Q are externally applied forces and charges.

3. COMBINING WAVEGUIDE AND CONVENTIONAL FE MODELS

The approach adopted was to use the waveguide finite element model to calculate the receptance of the waveguide to point forces. The receptance is used as a boundary condition, representing the waveguide in a conventional finite element model of the piezoelectric patch actuator. The forces applied to the waveguide are computed and then applied to the waveguide model to compute the response of the waveguide.

The forced response of the waveguide finite element model was developed by Damljanović and Weaver [10]. Their finite element formulation is slightly different to that used here but the method of solving the forced response still applies. The equations of motion may be solved by a method similar to that used for solving multi-degree-of-freedom damped oscillator systems. Equation 5 is complemented with an identity as follows,

$$\begin{bmatrix} K_0 - \omega^2 M & 0 \\ 0 & -K_2 \end{bmatrix} \begin{Bmatrix} u \\ \kappa u \end{Bmatrix} + \kappa \begin{bmatrix} K_1 & K_2 \\ K_2 & 0 \end{bmatrix} \begin{Bmatrix} u \\ \kappa u \end{Bmatrix} = \begin{Bmatrix} f \\ 0 \end{Bmatrix} \tag{8}$$

so that it may be written in the form

$$A\bar{u} - \kappa B\bar{u} = \bar{f}\,. \tag{9}$$

The free vibration problem (f = 0) is solved by performing an eigensolution. If a constant, real wavenumber (κ) is selected a real eigenproblem must be solved for the frequencies (ω) and mode shapes (ψ) of the propagating waves that correspond to this wavenumber. If the frequency is specified a complex eigenproblem must be solved. The wavenumbers that are obtained, by solving this problem, can be real, imaginary or complex and occur in pairs with opposite sign corresponding to waves travelling in opposite directions. If the number of nodes in the model is denoted N, the eigensolution results in $6N$ eigenvalue-eigenvector pairs κ_r and ψ_r.

The solution to the forced vibration problem is found by applying a Fourier transform to equation 9 to obtain an equation in the wavenumber domain, solution in the wavenumber domain and inverse Fourier transform to obtain the solution in the spatial domain [10],

$$\bar{u}(z) = -j\sum_{r=1}^{3N} \psi_r \frac{\psi_r{}^T \bar{f}(z)}{\psi_r{}^T B \psi_r} e^{-j\kappa_r z} \tag{10}$$

where the summation is performed only over the positive real poles, negative imaginary poles and complex poles with negative imaginary parts.

The response of the waveguide to forces at each of the degrees of freedom (dof) in contact with the piezoelectric patch may be computed by equation 10 and the receptance r_{ij} is defined as the response at dof *i* due to a unit force applied at dof *j,* i.e. $u_i = r_{ij} f_j$. The displacements at the interface dof (u_{in}) due to loads at the interface dof can then be related by the receptance matrix,

$$u_{in} = R f_{in} . \tag{11}$$

Recalling that the response of the waveguide was computed for a particular frequency of harmonic excitation, the inverse of this matrix (D_w) is the dynamic stiffness matrix of the interface dof's, at this frequency, i.e.

$$D_w u_{in} = f_{in} . \tag{12}$$

The dynamic stiffness matrix of the waveguide is symmetric but fully populated. It is also complex representing the mass/stiffness loading and the damping due to energy being radiated along the waveguide.

A dynamic stiffness for the piezoelectric patch can be computed at the excitation frequency and this matrix (D_p) can be partitioned into degrees of freedom in contact with the patch (u_{in}) and degrees of freedom not in contact (u_n). The two dynamic stiffness matrices can then be combined to represent the piezoelectric patch attached to the waveguide.

$$\begin{bmatrix} D_{p_{nn}} & D_{p_{nin}} \\ D_{p_{inn}} & D_{p_{inin}} + D_w \end{bmatrix} \begin{Bmatrix} u_n \\ u_{in} \end{Bmatrix} = \begin{Bmatrix} f_n \\ f_{in} \end{Bmatrix} \tag{13}$$

The forces in this equation include electrically induced piezoelectric forces. This equation allows the computation of the forced harmonic response of the piezoelectric patch (attached to the waveguide). The forces applied to the waveguide can be computed by substituting the interface displacements into equation 12. The response of the waveguide can then be computed by substituting these interface forces into equation 10. Our interest will often be in the amplitude of response of a particular mode of wave propagation rather than the amplitude at a specific point on the waveguide. This response is written as,

$$p_r(z) = -j \frac{\psi_r^T \bar{f}}{\psi_r^T B \psi_r} e^{-j\kappa_r z} . \tag{14}$$

4. RESULTS

Excitation of longitudinal and bending waves in infinite beams was analysed by Gibbs and Fuller [2, 3]. This analytical approximation is valid when the piezoelectric patch is thin compared to the beam and the cross section of the beam remains planar and perpendicular to the middle surface during deformation. The longitudinal (*u*) and bending (*w*) deformations due to two piezoelectric patches, one on either side, are:

$$u(x,t) = \frac{jK^L d_{31}(V_1 + V_2)}{2k_L h_a} \left(1 - e^{-jk_L L_a}\right) e^{j\omega t - jk_L x} \tag{15}$$

$$w(x,t) = \frac{d_{31} K^f (V_1 - V_2)}{4k_f^2 h_a} \left[\left(1 - e^{k_f L_a}\right) e^{-k_f x} - \left(1 - e^{-jk_f L_a}\right) e^{-jk_f x}\right] e^{j\omega t} \tag{16}$$

A 3mm thick aluminium beam with 2mm long 0.05mm thick PZT4 patches was analysed. The patches were driven by 1 kHz 1 Volt sinusoidal signals either in-phase or in opposite phase to excite the two waves. The finite element technique includes strain across the width of the beam, which is not included in the analytical approximation, therefore a few different beam widths were considered. The results of this analysis are listed in table 1. The response to the bending excitation is divided into the propagating wave (w_p) and the evanescent wave (w_e).

Table 1: Comparison of FE Results against analytical results for rectangular beam.

	Analytical	FE 0.4 mm wide	FE 4 mm wide	FE 40 mm Wide
k_l (1/m)	1.23	1.23	1.23	1.23
k_f (1/m)	37.75	36.89	36.24	35.11
u (m)	1.37×10^{-10} $-j\, 1.69 \times 10^{-13}$	1.53×10^{-10} $-j\, 1.80 \times 10^{-13}$	1.74×10^{-10} $-j\, 2.03 \times 10^{-13}$	1.78×10^{-10} $-j\, 2.07 \times 10^{-13}$
w_p (m)	1.39×10^{-10} $+j\, 3.69 \times 10^{-9}$	1.23×10^{-10} $+j\, 3.39 \times 10^{-9}$	1.32×10^{-10} $+j\, 3.68 \times 10^{-9}$	0.97×10^{-10} $+j\, 2.74 \times 10^{-9}$
w_e (m)	3.83×10^{-9}	-3.25×10^{-9} $-j\, 1.30 \times 10^{-11}$	-3.54×10^{-9} $-j\, 1.36 \times 10^{-11}$	-2.64×10^{-9} $-j\, 0.80 \times 10^{-11}$

While the results are not identical they are judged to be in sufficient agreement to conclude that the finite element method is correctly formulated and implemented.

The method developed in this paper is intended for the analysis of more complex waves in waveguides of complex cross-section. The rail cross-section shown in figure 1 is an example of such a waveguide. The rail cross-section has been divided into waveguide finite elements two-dimensional elements) and a piezoelectric patch is attached to the web of the rail. The patch was modelled with three-dimensional piezoelectric elements, extending in the direction along the rail axis. The patch was 2mm thick, 30 mm long and extended 10 mm along the rail. The waveguide model can be used to compute the dispersion curves of the rail. Figure 2 shows the wavenumber – frequency relation for the first 30 propagating waves in the rail. In addition to the propagating waves there are also evanescent, or near-field, waves, which decay exponentially along the waveguide. The propagating waves at a frequency of 60 kHz can be very complex in nature. Two examples are shown in figure 3.

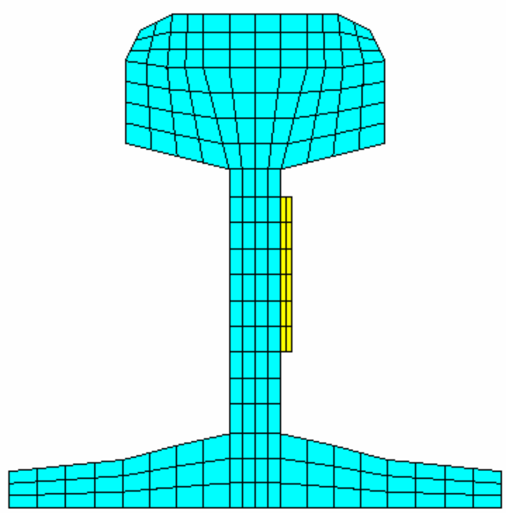

Figure 1. Rail geometry and piezoelectric patch attachment.

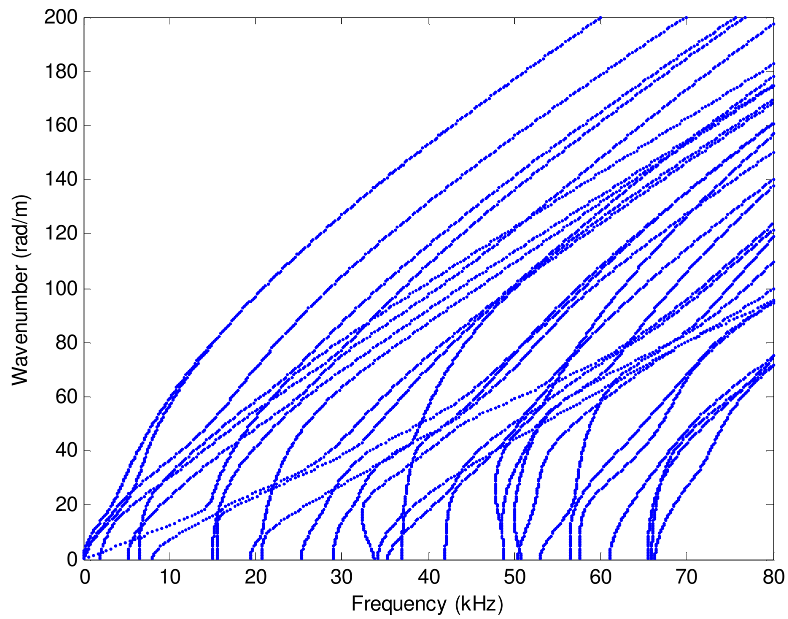

Figure 2. Wavenumber – Frequency dispersion curves for the 1st 30 the propagating waves.

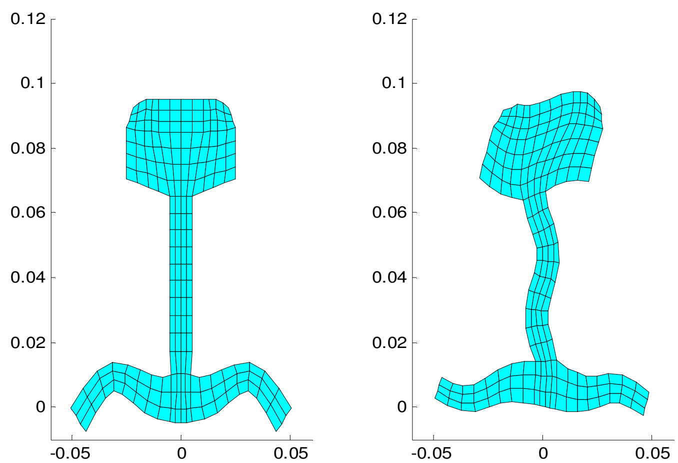

Figure 3. Propagating waves at 60 kHz with wavenumbers of 127 1/m (left) and 54 1/m (right).

The patch, attached to the rail, was excited with a 1 Volt amplitude sinusoidal signal at 60 kHz. A second patch was added on the opposite side of the rail web. It is expected that by exciting the two patches in-phase the 'symmetric' mode would be strongly excited, while excitation in opposite phase would strongly excite the 'bending' wave. The amplitudes of these two modes of wave propagation for these cases are compared to the single patch case in table 2. The mode amplitudes correspond to the largest displacement of a nodal degree of freedom. The results indicate that a mode can be excited more than twice as effectively by using two patches on opposite sides. It is believed that this is because the induced strain available from the patches is used to mainly excite the one mode and is not 'wasted' on the other. It is clear that using two patches on opposite sides of the web offers important advantages for exciting a particular mode and this result could be generalised to more patches around the rail cross-section and along the rail.

Table 2: Rail Response to Different Patch Configurations.

Configuration	'Symmetric' Mode Response (m)	'Bending' Mode Response (m)
1 Patch	$6.23+j8.79x10^{-10}$	$1.58+j5.65x10^{-10}$
2 Patches – in-phase	$2.44+j3.48x10^{-9}$	$1.01+j0.18x10^{-12}$
2 Patches – out of phase	$3.48+j0.45x10^{-12}$	$6.90+j23.4x10^{-10}$

5. CONCLUSIONS

A numerical method for modelling the excitation of waves in constant cross-section waveguides, by piezoelectric patch actuators was developed. The method combined conventional three-dimensional finite elements representing the piezoelectric patch actuator with special two-dimensional finite elements representing the infinite waveguide. The case of two piezoelectric patches on either side of an infinite beam was analysed and compared to an approximate analytical solution to verify the numerical method. The method was then applied to the excitation of a rail cross-section waveguide at high frequency where the modes of propagation are not simple. The model demonstrated the interesting possibilities available for excitation of specific waves by multiple patches. The optimisation of patches and arrays of patches for exciting individual waves will be considered in future.

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