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ON DYNAMICS AND CONTROL OF VIBRATORY GYROSCOPES WITH SPHERICAL SYMMETRY

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Abstract

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It was shown in 1985 by Acad. V. Zhuravlev that the angular rate of a pure vibrating mode excited in a vibratory gyroscope with spherical symmetry is proportional to an inertial angular rate of the gyroscope. The effect is three dimensional and hence, it could be potentially used as a conception of a spatial inertial rotational sensor. Furthermore these effects are important in acoustics, geophysics and astrophysics. The effect was investigated qualitatively without specifying of a coordinate system and determination of the scale factors. In the present paper the effect of vibrating pattern precession is considered in a spherical coordinates. On the basis of exact solution of 3-D equations of motion of thick isotropic sphere, which are obtained in the spherical Bessel and the associated Legendre functions, the effects of rotation are investigated and scales factors are determined for different vibrating modes of the spherical body, spheroidal and torsional. Corresponding scales factors are calculated depending on nature of vibrating modes and their number. For realization of a three axes sensor it is necessary to realize three orthogonal spherical coordinate systems. Elements of control of vibrating spherically symmetric body are considered and possible imperfections are discussed.

Generating equations of motion of the spherical body ($\Omega = 0$) are[1]:

$$\begin{cases} \dfrac{\partial\, \sigma_{rr}}{\partial r} + \dfrac{1}{r}\dfrac{\partial\, \sigma_{r\theta}}{\partial \theta} + \dfrac{1}{r\sin\theta}\dfrac{\partial\, \sigma_{r\varphi}}{\partial \varphi} + \dfrac{2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\varphi\varphi} + \cot\theta\cdot\sigma_{r\theta}}{r} = \rho\dfrac{\partial^2 w}{\partial t^2} \\ \dfrac{1}{r}\dfrac{\partial\, \sigma_{\theta\theta}}{\partial \theta} + \dfrac{\partial\, \sigma_{r\theta}}{\partial r} + \dfrac{1}{r\sin\theta}\dfrac{\partial\, \sigma_{\theta\varphi}}{\partial \varphi} + \dfrac{3\sigma_{r\theta} + \cot\theta\cdot\left(\sigma_{\theta\theta} - \sigma_{\varphi\varphi}\right)}{r} = \rho\dfrac{\partial^2 u}{\partial t^2} \\ \dfrac{1}{r\sin\theta}\dfrac{\partial\, \sigma_{\varphi\varphi}}{\partial \varphi} + \dfrac{\partial\, \sigma_{r\varphi}}{\partial r} + \dfrac{1}{r}\dfrac{\partial\, \sigma_{\theta\varphi}}{\partial \theta} + \dfrac{3\sigma_{r\varphi} + 2\cot\theta\cdot\sigma_{\theta\varphi}}{r} = \rho\dfrac{\partial^2 v}{\partial t^2} \end{cases} \quad \textbf{(1)}$$

where the stresses: $\sigma_{rr} = \lambda\left(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{\varphi\varphi}\right) + 2\mu\varepsilon_{rr}$; $\sigma_{\theta\theta} = \lambda\left(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{\varphi\varphi}\right) + 2\mu\varepsilon_{\theta\theta}$; $\sigma_{\varphi\varphi} = \lambda\left(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{\varphi\varphi}\right) + 2\mu\varepsilon_{\varphi\varphi}$; $\sigma_{r\theta} = \mu\varepsilon_{r\theta}$; $\sigma_{r\varphi} = \mu\varepsilon_{r\varphi}$; $\sigma_{\theta\varphi} = \mu\varepsilon_{\theta\varphi}$ and strains are: $\varepsilon_{rr} = w'_r$; $\varepsilon_{\theta\theta} = \dfrac{1}{r}(u'_\theta + w)$; $\varepsilon_{\varphi\varphi} = \dfrac{1}{r}\left(\cot\theta\cdot u + \dfrac{1}{\sin\theta}v'_\varphi + w\right)$; $\varepsilon_{r\theta} = u'_r + \dfrac{1}{r}(-u + w'_\theta)$;

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$\varepsilon_{r\varphi} = v'_r + \frac{1}{r}\left(-v + \frac{1}{\sin\theta} w'_\varphi\right)$; $\varepsilon_{\theta\varphi} = \frac{1}{r}\left(\frac{1}{\sin\theta} u'_\varphi - \cot\theta \cdot v + v'_\theta\right)$. By means of change of variables $(u,v,w) \to (\Phi,\Psi,\mathrm{X})$: $w = \Phi'_r + r\cdot\left[\left(\mathrm{X}''_{rr} + \frac{2}{r}\mathrm{X}'_r\right) - \nabla^2\mathrm{X}\right]$; $u = \left[\mathrm{X}'_r + \frac{1}{r}(\Phi+\mathrm{X})\right]'_\theta + \frac{1}{a\sin\theta}\Psi'_\varphi$;

$v = \frac{1}{\sin\theta}\cdot\left[\mathrm{X}'_r + \frac{1}{r}(\Phi+\mathrm{X})\right]'_\varphi - \frac{1}{a}\Psi'_\theta$ the variables are separated:

$$(\lambda+2\mu)\cdot\nabla^2\Phi = \rho\ddot{\Phi}; \qquad \mu\cdot\nabla^2\Psi = \rho\ddot{\Psi}; \qquad \mu\cdot\nabla^2\mathrm{X} = \rho\ddot{\mathrm{X}} \tag{2}$$

and solution could be represented as a sum of *mn* – spherical harmonics:

$$\begin{cases} \Phi_{mn}(r,\theta,\varphi) = A_{mn}\cdot j_n(k_1 r)\cdot P_n^m(\cos\theta)\cdot\cos(m\varphi) \\ \mathrm{X}_{mn}(r,\theta,\varphi) = B_{mn}\cdot j_n(k_2 r)\cdot P_n^m(\cos\theta)\cdot\cos(m\varphi) \\ \Psi_{mn}(r,\theta,\varphi) = D_{mn}\cdot j_n(k_2 r)\cdot P_n^m(\cos\theta)\cdot\sin(m\varphi) \end{cases} \tag{3}$$

where the wave numbers $k_1 = k_1(\omega) = \omega/c_1$; $k_2 = k_2(\omega) = \omega/c_2$ and $c_1 = \sqrt{(\lambda+2\mu)/\rho}$; $c_2 = \sqrt{\mu/\rho}$ - speeds of extensional and inextensional waves propagation. Due to absence of radial and tangential stresses on the spherical surface ($r = a$) the boundary conditions are:

$$[\sigma_{rr}]_{r=a} = [\sigma_{r\theta}]_{r=a} = [\sigma_{r\varphi}]_{r=a} = 0 \tag{4}$$

Angular rate of the vibrating pattern precess in the rotating coordinate system is:

$$\dot{\psi} = -\frac{2\int_0^\pi\int_0^a\left[\left(U(r,\theta)\cdot\cos\theta + W(r,\theta)\cdot\sin\theta\right)\cdot V(r,\theta)\right]\cdot r^2\cdot\sin\theta\, dr\, d\theta}{\int_0^\pi\int_0^a\left[U^2(r,\theta) + V^2(r,\theta) + W^2(r,\theta)\right]\cdot r^2\cdot\sin\theta\, dr\, d\theta}\cdot\Omega \tag{5}$$

Example. Let us consider an example of a sphere of radius $a = 0.5\,m$ made from an aluminium alloy with modulus of elasticity $E = 7\cdot10^{10}\ N/m^2$, Poisson's ration $\nu = 0.33$ and mass density $\rho = 2.7\cdot10^3\ kg/m^3$. Calculated real values of eigenvalues in Hz of spheroidal modes are given in the table for the cases of free outer surface and acoustically loaded surface for $n = 2, m = 2;\quad n = 3, m = 2;\quad n = 3, m = 3$. Corresponding values of the Bryan's factors of the spheroidal modes are also given in the table:

	m=n=2			m=2, n=3			m=n=3		
Eigenvalues (Hz)	2633	5056	8563	3924	6654	9910	3924	6654	9910
Bryan's factors	0.921	0.137	0.300	0.515	0.127	0.136	0.634	0.000	0.073

Principles of excitation and control of vibrating patterns of the spherical vibratory gyroscopes are similar to those of the well known hemispherical rate sensors.

Conclusions

1. Expression for Bryan's factor was derived, which characterizes the coefficients of proportionality between angular rate of precession of a vibrating pattern and the inertial angular rate of the spherical isotropic elastic bod.
2. It was establish out that the Bryan's factor is an invariant of sphere's radius, its mass density and modulus of elasticity; it depends on Poisson's ratio.
3. It was found that in the case of spheroidal oscillations the Bryan's factor of radiated body is higher than the value of these factor for free body of the same mode; torsional oscillations do not interact with an ideal non-viscous acoustic medium.

References

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