A NOVEL APPROACH TO THE HELMHOLTZ INTEGRAL EQUATION SOLUTION BY FOURIER SERIES EXPANSION FOR ACOUSTIC RADIATION AND SCATTERING PROBLEMS

Michael Shatalov, Igor Fedotov, Stephan Joubert

Sensor Science and Technology (SST)
of CSIR Material Science and Manufacturing (MSM),

Department of Mathematical Technology
Tshwane University of Technology, Pretoria, South Africa

Days on Diffraction
(St.Petersburg, May 30 – June 02, 2006)

Abstract

Conventional methods of solution of the Helmholtz integral equation consist in discretization of a radiating/scattering boundary on multiple boundary elements, assumption of a smooth distribution of the velocity potential on every element and transforming of the original problem to a system of linear algebraic equations. This method needs a large amount of time consuming calculations in the case when the radiating/scattering surface is large and the frequency of the acoustic field could not be considered as low or high and belongs to an "intermediate" frequency range. Analysis of these effects is practically impossible on a conventional PC due to the number of boundary elements, which are necessary to spread over the surface of the large-scale structure to guarantee the numerical accuracy of solution. In the present paper we propose to use a novel method of solution of the Helmholtz integral equation, which is based on expansion of the integrands in double Fourier series. The main difficulty of realization of the Fourier series approach is that the kernels of this equation do not satisfy to the Dirichlet's theorem and hence, could not be directly expanded into Fourier series. To overcome the abovementioned difficulty we represent the Helmholtz integral as sum of the integral with modified kernel, which satisfy the Dirichlet's theorem and so could be expanded in the Fourier series, and an additional integral in the vicinity of the point of singularity. This approach helps to substantially reduce the volume of calculations, takes advantage of fast discrete Fourier transformations and achieves a substantial progress in solution of acoustic radiation and scattering problems. The typical example of scattering of an obliquely incident plane wave by a large-scale structure composed by a cylinder with two hemispherical end caps is considered.

CONTENT

- INTRODUCTION
- IDEA OF THE METHOD
- FORMULATION IN SPHERICAL COORDINATES
- EXAMPLES
- CONCLUSIONS

CONTENT

- **INTRODUCTION**
- **IDEA OF THE METHOD**
- **FORMULATION IN SPHERICAL COORDINATES**
- **EXAMPLES**
- **CONCLUSIONS**

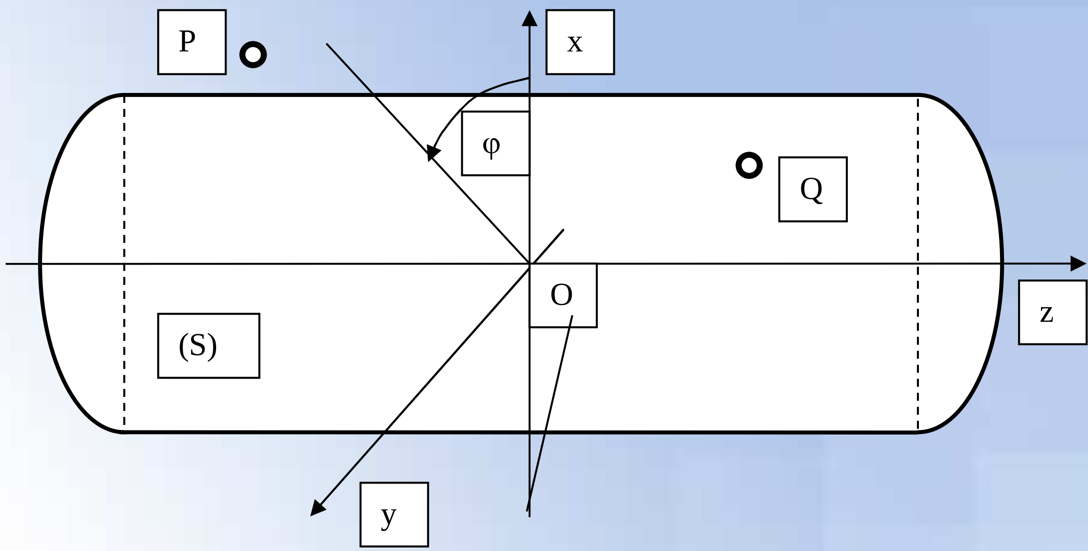
INTRODUCTION

Combined Helmholtz integral equation formulation is often used in boundary element methods for description of the effects of radiation and scattering of physical fields, for example, acoustical fields. This method has a substantial advantage over the “domain” methods, such as finite element methods, due to reduction of three dimensional problems to two dimensions. Seybert described the method of integral equations for solution of radiation and scattering problems for axisymmetric bodies and boundary conditions. Further Soenarko generalized this method to the problems with axisymmetric bodies and arbitrary boundary conditions. Present paper considers a general case of arbitrary body, which could be uniquely characterized by a system of two parameters, with arbitrary boundary conditions. After formulation of idea of the method the algorithm is formulated and numerical example of a plane acoustic wave scattering by a cylindrical body with spherical end caps is considered.

CONTENT

- INTRODUCTION
- **IDEA OF THE METHOD**
- FORMULATION IN SPHERICAL COORDINATES
- EXAMPLES
- CONCLUSIONS

IDEA OF THE METHOD - 1



$$C(P)\cdot\Phi(P)=\int_{(A)}\left[\frac{\partial\ g(P,Q)}{\partial n_Q}\cdot\Phi(Q)+g(P,Q)\cdot V_n(Q)\right]dA+4\pi\cdot\Psi^{(i)}(P)$$

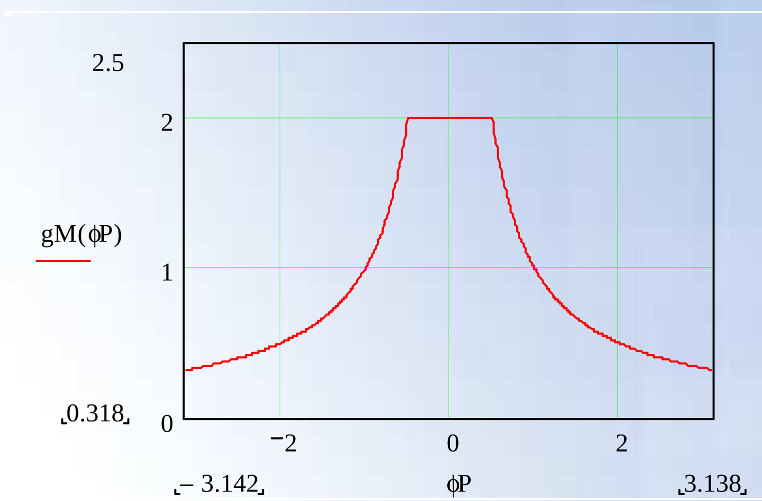
$$g(P,Q)=\frac{e^{-i\,k\,\rho(P,Q)}}{\rho(P,Q)} \qquad k=\omega/c \qquad C(P)=4\pi+\int_{(A)}\frac{\partial}{\partial n_Q}\left[\frac{1}{\rho(P,Q)}\right]dA$$

IDEA OF THE METHOD - 2

$$C(P)\cdot\Phi(P)=\int\limits_{(A)}\left[\frac{\partial\ g(P,Q)}{\partial n_Q}\cdot\Phi(Q)+g(P,Q)\cdot V_n(Q)\right]dA+4\pi\cdot\Psi^{(i)}(P)\approx$$

$$\approx\int\limits_{(A)}\left[\frac{\partial\ \overset{\square}{g}(P,Q)}{\partial n_Q}\cdot\Phi(Q)+\overset{\square}{g}(P,Q)\cdot V_n(Q)\right]dA+$$

$$\Phi(P)\cdot\int\limits_{(\Delta A)}\frac{\partial\ g(P,Q)}{\partial n_Q}\,d(\Delta A)+V_n(P)\cdot\int\limits_{(\Delta A)}g(P,Q)\,d(\Delta A)+4\pi\cdot\Psi^{(i)}(P)$$



IDEA OF THE METHOD - 3

$$C(P)\cdot\Phi(P)\approx\int_{(A)}\left[\frac{\partial g(P,Q)}{\partial n_Q}\cdot\Phi(Q)+g(P,Q)\cdot V_n(Q)\right]dA+V_n(P)\cdot\int_{(\Delta A)}g(P,Q)\,d(\Delta A)+4\pi\cdot\Psi^{(i)}(P)$$

$$C(P)=4\pi+\int_{(A)}\frac{\partial}{\partial n_Q}\left[\frac{1}{\rho(P,Q)}\right]dA-\int_{(\Delta A)}\frac{\partial g(P,Q)}{\partial n_Q}\,d(\Delta A)$$

If $V_n(Q)$ - prescribed:

$$C(P)\cdot\Phi(P)\approx\int_{(A)}\frac{\partial g(P,Q)}{\partial n_Q}\cdot\Phi(Q)\,dA+\left\{\int_{(A)}g(P,Q)\cdot V_n(Q)\,dA+V_n(P)\cdot\int_{(\Delta A)}g(P,Q)\,d(\Delta A)+4\pi\cdot\Psi^{(i)}(P)\right\}$$

CONTENT

- INTRODUCTION
- IDEA OF THE METHOD
- **FORMULATION IN SPHERICAL COORDINATES**
- EXAMPLES
- CONCLUSIONS

FORMULATION IN SPHERICAL COORDINATES - 1

$$C(P)\cdot\Phi(P) \approx \int_{(A)} \frac{\partial g(P,Q)}{\partial n_Q}\cdot\Phi(Q)\,dA + \left\{\int_{(A)} g(P,Q)\cdot V_n(Q)\,dA + V_n(P)\cdot\int_{(\Delta A)} g(P,Q)\,d(\Delta A) + 4\pi\cdot\Psi^{(i)}(P)\right\}$$

In Spherical Coordinates:

$$\int_{(A)} \Psi\left(\theta_Q,\varphi_Q\right)dA = \int_0^{2\pi}\int_0^{\pi}\Psi\left(\theta_Q,\varphi_Q\right)\cdot\frac{r^2\left(\theta_Q,\varphi_Q\right)}{\cos\left[\overrightarrow{r\left(\theta_Q,\varphi_Q\right)},\overrightarrow{n_Q}\right]}\cdot\sin\theta_Q\,d\theta_Q\,d\varphi_Q$$

Truncated Fourier Series Representation of Solution:

$$\Phi(\theta,\varphi) \approx a_0 + \sum_{n=1}^{N}\sum_{m=1}^{M}\left[a_{mn}\cos(m\varphi) + b_{mn}\sin(m\varphi)\right]\cdot\cos(n\theta)$$

FORMULATION IN SPHERICAL COORDINATES - 2

$$C(P)\cdot\Phi(P)\approx\int_{(A)}\frac{\partial g(P,Q)}{\partial n_Q}\cdot\Phi(Q)\,dA+\left\{\int_{(A)} g(P,Q)\cdot V_n(Q)\,dA+V_n(P)\cdot\int_{(\Delta A)} g(P,Q)\,d(\Delta A)+4\pi\cdot\Psi^{(i)}(P)\right\}$$

$$\int_{(A)}\Psi(\theta_Q,\varphi_Q)\,dA=\int_0^{2\pi}\int_0^{\pi}\Psi(\theta_Q,\varphi_Q)\cdot\frac{r^2(\theta_Q,\varphi_Q)}{\cos\left[\overrightarrow{r(\theta_Q,\varphi_Q)},\overrightarrow{n_Q}\right]}\cdot\sin\theta_Q\,d\theta_Q\,d\varphi_Q$$

$$f_1(\theta_P,\varphi_P,\theta_Q,\varphi_Q)=\frac{\partial g(\theta_P,\varphi_P,\theta_Q,\varphi_Q)}{\partial n_Q}\cdot\frac{r^2(\theta_Q,\varphi_Q)}{\cos\left[\overrightarrow{r(\theta_Q,\varphi_Q)},\overrightarrow{n_Q}\right]}\cdot\sin\theta_Q \qquad f_2(\theta_P,\varphi_P,\theta_Q,\varphi_Q)=g(\theta_P,\varphi_P,\theta_Q,\varphi_Q)\cdot\frac{r^2(\theta_Q,\varphi_Q)}{\cos\left[\overrightarrow{r(\theta_Q,\varphi_Q)},\overrightarrow{n_Q}\right]}\cdot\sin\theta_Q$$

$$F_1(\theta_P,\varphi_P,\theta_Q,\varphi_Q)=\frac{1}{2}\cdot\begin{cases} f_1(\theta_P,\varphi_P,\theta_Q,\varphi_Q), & if\ \theta_Q\in[0,\pi] \\ f_1(2\pi-\theta_P,\varphi_P,2\pi-\theta_Q,\varphi_Q), & if\ \theta_Q\in(\pi,2\pi)\end{cases} \qquad F_2(\theta_P,\varphi_P,\theta_Q,\varphi_Q)=\frac{1}{2}\cdot\begin{cases} f_2(\theta_P,\varphi_P,\theta_Q,\varphi_Q), & if\ \theta_Q\in[0,\pi] \\ f_2(2\pi-\theta_P,\varphi_P,2\pi-\theta_Q,\varphi_Q), & if\ \theta_Q\in(\pi,2\pi)\end{cases}$$

$$F_1(\theta_P,\varphi_P,\theta_Q,\varphi_Q)\approx c_0(P)+\sum_{n=1}^{N_1}\sum_{m=1}^{M_1}\left[c_{mn}(\theta_P,\varphi_P)\cos(m\varphi_Q)+d_{mn}(\theta_P,\varphi_P)\sin(m\varphi_Q)\right]\cdot\cos(n\theta_Q)$$

$$F_2(\theta_P,\varphi_P,\theta_Q,\varphi_Q)\approx f_0(P)+\sum_{n=1}^{N_1}\sum_{m=1}^{M_1}\left[f_{mn}(\theta_P,\varphi_P)\cos(m\varphi_Q)+h_{mn}(\theta_P,\varphi_P)\sin(m\varphi_Q)\right]\cdot\cos(n\theta_Q)$$

FORMULATION IN SPHERICAL COORDINATES - 3

$$\tilde{V}_n(Q) = \frac{1}{2} \cdot \begin{cases} V_n(\theta_Q, \varphi_Q), & if\ \theta_Q \in [0, \pi] \\ V_n(2\pi - \theta_Q, \varphi_Q), & if\ \theta_Q \in (\pi, 2\pi) \end{cases} \approx V_0 + \sum_{n=1}^{N_1} \sum_{m=1}^{M_1} \left[V_{mn}^{(c)} \cos(m\varphi_Q) + V_{mn}^{(s)} \sin(m\varphi_Q) \right] \cdot \cos(n\theta_Q)$$

$$\tilde{C}(\theta_P, \varphi_P) \cdot \Phi(\theta_P, \varphi_P) \approx \int_0^{2\pi} \int_0^{2\pi} \left[F_1(\theta_P, \varphi_P, \theta_Q, \varphi_Q) \cdot \Phi(\theta_Q, \varphi_Q) + F_2(\theta_P, \varphi_P, \theta_Q, \varphi_Q) \cdot \tilde{V}_n(\theta_Q, \varphi_Q) \right] d\theta_Q\, d\varphi_Q$$

$$+ V_n(\theta_P, \varphi_P) \cdot \int_{\varphi_P - \Delta\varphi}^{\varphi_P + \Delta\varphi} \int_{\theta_P - \Delta\theta}^{\theta_P + \Delta\theta} g(\theta_P, \varphi_P, \theta_Q, \varphi_Q) \cdot \frac{r^2(\theta_Q, \varphi_Q)}{\cos\left[\overrightarrow{r(\theta_Q, \varphi_Q)}, \overrightarrow{n_Q} \right]} \cdot \sin\theta_Q\, d\theta_Q\, d\varphi_Q + 4\pi \cdot \Psi^{(i)}(\theta_P, \varphi_P)$$

$$\tilde{C}(\theta_P, \varphi_P) = C(\theta_P, \varphi_P) - \int_{\varphi_P - \Delta\varphi}^{\varphi_P + \Delta\varphi} \int_{\theta_P - \Delta\theta}^{\theta_P + \Delta\theta} \frac{\partial\, g(\theta_P, \varphi_P, \theta_Q, \varphi_Q)}{\partial n_Q} \cdot \frac{r^2(\theta_Q, \varphi_Q)}{\cos\left[\overrightarrow{r(\theta_Q, \varphi_Q)}, \overrightarrow{n_Q} \right]} \cdot \sin\theta_Q\, d\theta_Q\, d\varphi_Q$$

$$g = e^{-ik\rho} \big/ \rho$$

$$\rho = \sqrt{r^2(\theta_Q, \varphi_Q) + r^2(\theta_P, \varphi_p) - 2 \cdot r(\theta_Q, \varphi_Q) \cdot r(\theta_P, \varphi_p) \cdot \left[\cos\theta_Q \cos\theta_P + \sin\theta_Q \sin\theta_P \cos(\varphi_Q - \varphi_p) \right]}$$

$$\frac{\partial\, g}{\partial n_Q} = \frac{\partial}{\partial \rho} \left(\frac{e^{-ik\rho}}{\rho} \right) \cdot \frac{\partial\, \rho}{\partial n_Q} = -ik \cdot \left\{ g \cdot \cos\left[(\overrightarrow{P,Q}), \overrightarrow{n_Q} \right] \right\} - g \cdot \left\{ \frac{1}{\rho} \cdot \cos\left[(\overrightarrow{P,Q}), \overrightarrow{n_Q} \right] \right\}$$

FORMULATION IN SPHERICAL COORDINATES - 4

$$p_0(P)\cdot a_0+\sum_{n=1}^{N_1}\sum_{m=1}^{M_1}\left[p_{mn}(P)\cdot a_{mn}+q_{mn}(P)\cdot b_{mn}\right]\approx r(P)$$

$$p_0(P)=\tilde{C}(P)-4\pi^2 c_0$$

$$p_{mn}(P)=\tilde{C}(P)\cos(m\varphi_P)\cos(n\theta_P)-\pi^2 c_{mn}$$

$$q_{mn}(P)=\tilde{C}(P)\sin(m\varphi_P)\cos(n\theta_P)-\pi^2 d_{mn}$$

$$r(P)=\pi^2\left\{4f_0(P)V_0(P)+\sum_{n=1}^{N_1}\sum_{m=1}^{M_1}\left[f_{mn}(P)\cdot V_{mn}^{(c)}(P)+h_{mn}(P)\cdot V_{mn}^{(s)}(P)\right]\right\}$$

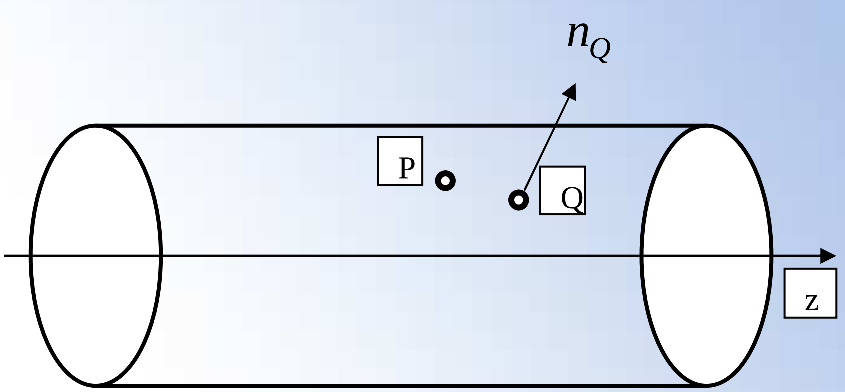
$$+V_n(P)\cdot\int_{\varphi_P-\Delta\varphi}^{\varphi_P+\Delta\varphi}\int_{\theta_P-\Delta\theta}^{\theta_P+\Delta\theta} g(P,Q)\cdot\frac{r^2(Q)}{\cos\left[\overrightarrow{r(Q)},\overrightarrow{n_Q}\right]}\cdot\sin\theta_Q\, d\theta_Q\, d\varphi_Q+4\pi\Psi^{(i)}(P)$$

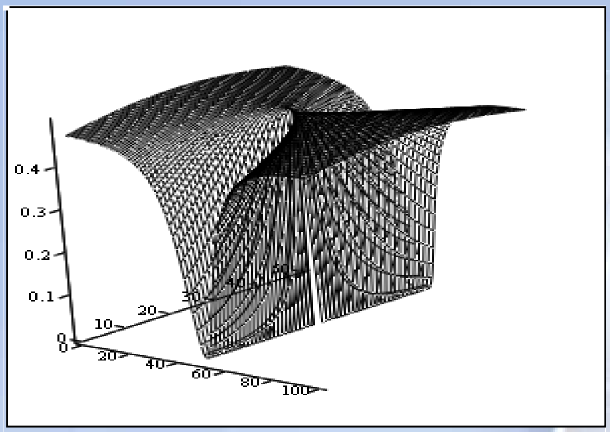
CONTENT

- INTRODUCTION
- IDEA OF THE METHOD
- FORMULATION IN SPHERICAL COORDINATES
- **EXAMPLES**
- CONCLUSIONS

EXAMPLE - 1

Discontinuity of $\left(\frac{1}{\rho}\cdot\frac{\partial\ \rho}{\partial n_Q}\right)$





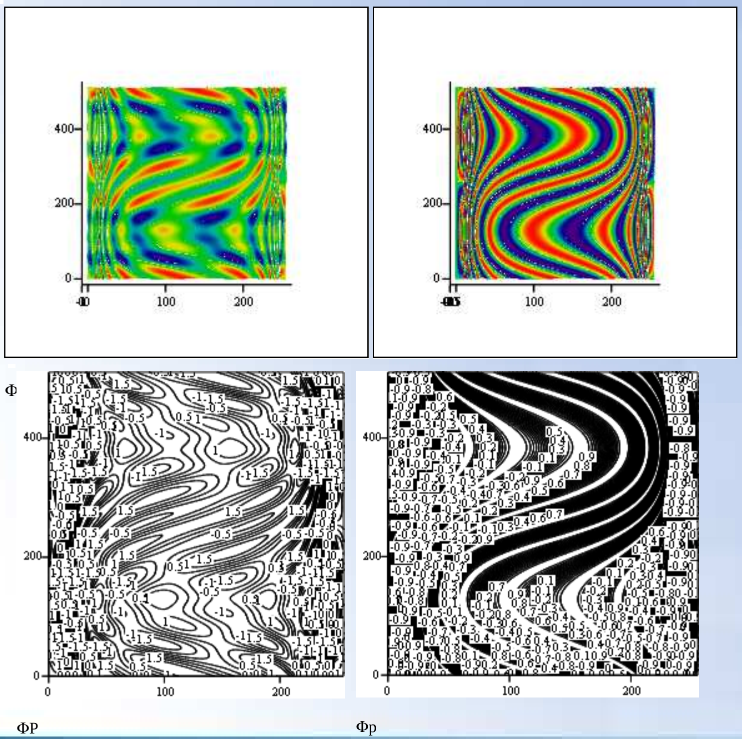
DRN

$$\|\vec{\rho}\| = \sqrt{4r^2\sin^2\left(\frac{\varphi}{2}\right)+z^2}$$

$$\frac{1}{\rho}\frac{\partial\ \rho}{\partial n_Q} = \frac{1}{\|\vec{\rho}\|}\frac{\partial\ \rho}{\partial n_Q} = \frac{2r\cdot\sin^2\left(\frac{\varphi}{2}\right)}{4r^2\cdot\sin^2\left(\frac{\varphi}{2}\right)+z^2}$$

EXAMPLE - 2

$a = 3\ \ m$ $h = 54\ \ m$ $\alpha = 30\ \ deg$ $f = 500\ \ Hz$



CONTENT

CONCLUSIONS

The method of solution of the Helmholtz integral equation is formulated, which is based on representation of a velocity potential in terms of Fourier series and finding the Fourier coefficients of this expansion. The Green function is modified so to satisfy the Dirichlet's theorem. Fourier coefficients of the modified Green functions are calculated using a discrete Fourier transform, in particular case by a fast Fourier transformation. Using orthogonality of the sine and cosine functions the original problem is reduced to an overdetermined system of linear algebraic equations to obtain the unknown coefficients of the Fourier series expansion. This method is applicable to a broad range of acoustical problems of radiation and scattering. It can be easily parallelized and realized on grid or vector computers. The example of calculation of near acoustic fields of large-scale structures is given.