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Two-Stage Optimization in a Transportation Problem

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A study of the economic distribution of maize throughout South Africa is reported. Although the problem of minimizing total transportation costs in such a situation is a classical one, and its solution is well known, there was in this case a high degree of degeneracy in the system and thus the solution was not unique. Also, since a user is required to pay his own transportation costs, the various optimal solutions were not equivalent. A secondary problem thus arose, viz. that of selecting from these optimal solutions the one which would be fairest to all users. A heuristic and a goal programming method for solving this secondary problem are discussed.

INTRODUCTION

THE SITUATION discussed in this paper seems at first glance to be an example of the classical transportation problem. The commodity to be carried by rail is maize, and the supply and demand points are scattered throughout the country. The primary objective is normally to minimize the total rail cost (but alternatively minimization of the total distance travelled may be desired), but in addition it is also required to distribute the costs fairly amongst all users.

In South Africa, virtually all maize produced is sold in the first instance to a statutory body called the Maize Board, which receives it at storage depots distributed across the country. One of the functions of the Maize Board is to ensure that every year each farmer receives a fixed price which is sufficient to ensure that he will make a fair profit irrespective of current surpluses or deficits. The Board is then responsible for further marketing of the maize (at a price generally below the cost to the Board, made possible by government subsidy since maize is a staple food) and also for disposal of any surplus through exports.

The Board is thus effectively the only supplier of maize, but supplies are available at a number of different locations (about 300). Maize ordered by about 120 users (mills) is delivered f.o.r. (free on rail) by the Board at one or more of the supply locations as designated by it. (The Board does not operate any intermediate redistribution centres.) Subsequent rail costs, which are not in any way controllable by the Maize Board, are thus borne by the user.

By virtue of this monopolistic situation, decisions of the Maize Board dictate from which sources a given user is to obtain his supplies, and thus also the rail costs incurred by him. The Maize Board thus bears a major responsibility for ensuring that in making its allocations there is no discrimination in favour of one user at the expense of another. In particular, it is almost invariably true that if one user is allowed to obtain supplies at a cost which is minimal for him, this will be at the expense of one or more other users, a discriminatory situation which the Board cannot allow.

The responsibilities of the Board are thus two-fold, viz. it must ensure, first, that costs to all users are kept as low as possible, and secondly, that each user will bear a fair share of the unavoidable costs. It is this two-fold responsibility, and particularly the second, which gave rise to the study reported in this paper.

In order to clarify discussion of the problem and its solution, we shall refer repeatedly to the sample problem introduced in Table 1. This problem, involving 10 user and 30 supply points, has been abstracted from an actual practical situation. It illustrates most of the essential points, but owing to the complexity of the interactions which

TABLE 1. ILLUSTRATIVE EXAMPLE: SUPPLIES AND DEMANDS*

Supply point	Amount available in tons	Supply point	Amount available in tons
S1	2104	S16	4341
S2	3204	S17	1627
S3	1357	S18	2659
S4	2492	S19	6926
S5	2998	S20	1448
S6	2397	S21	3259
S7	1563	S22	2602
S8	7729	S23	3705
S9	3762	S24	3465
S10	2201	S25	2340
S11	2422	S26	2349
S12	1344	S27	1010
S13	3323	S28	3990
S14	432	S29	1925
S15	1936	S30	2463
		Total	83,373

User point	Amount required
D1	3168
D2	14,217
D3	9768
D4	11,538
D5	5255
D6	1975
D7	12,032
D8	7402
D9	700
D10	5750
Total	71,805

* Cost data are obtainable from the authors on request.

arise within the fullscale problem, it is unfortunately not possible on such a small scale to maintain complete realism, nor to represent the full power of the techniques.

In deciding from which supply points under its control the requirements of a given user are to be met, the Maize Board has two goals. The first goal is usually formulated as minimization of the total rail costs for delivery of all supplies, as this is considered to be in the best interest of the country as a whole (although in some studies the effect of minimizing total distance has been considered).

The attainment of this first goal is for convenience termed the "primary" optimization step. This step is in fact the classical transportation problem, although in practice some preliminary steps are required in setting up the cost tableau. These steps are necessary in that the computation of the rail cost (or distance) between any two points requires first the computation of the shortest path between these points through the rail network, itself a non-trivial problem. Once the cost tableau has been set up, the primary problem can be solved by making use of any standard transportation algorithm.[2]

Of importance in the rest of the paper is the definition of feasible arcs, viz. those supply–user routes for which $c_{ij} - u_i - v_j = 0$, where c_{ij} is the relevant cost and u_i, v_j are the optimal dual variables (cf. Simmonard[2]). Now any solution in which only these feasible arcs are used will give the optimal solution, i.e. the minimum total rail cost or minimum total distance travelled. (Note that this holds after an artificial source or destination is added to take up a deficit or surplus in the maize supply at zero cost.)

Using a transportation algorithm, the optimal solution to the sample problem can be obtained, giving 106 feasible arcs, even though a feasible, minimal-cost solution can be found using only 39 of these arcs. The particular 39 arcs chosen by a specific algorithm is characteristic of the code used, the choice being always somewhat arbitrary.

TABLE 2. UNIT COSTS FOR A PARTICULAR ("ARBITRARY") MINIMAL-COST SOLUTION TOGETHER WITH MINIMUM AND MAXIMUM UNIT COSTS WITHIN MINIMAL-COST SOLUTIONS

User point	Unit cost	Minimum possible cost	Maximum possible cost
D1	1935	1862	1975
D2	1556	1532	1706
D3	1633	1532	1706
D4	1908	1865	1911
D5	1897	1879	1973
D6	1939	1833	1989
D7	1982	1924	1982
D8	1969	1930	2013
D9	1534	1534	1534
D10	373	373	391

The costs per unit for each user for one such "arbitrary" (but still minimum-cost) solution are shown in Table 2, together with the minimum and maximum unit costs that each user could possibly sustain while still maintaining an overall-minimum total cost.

Note that users D2 and D3 have the same minimum unit costs and the same maximum unit costs. Since it is not possible for all users to realize their minimum costs, D2 and D3, if they are to bear a fair share of the total costs, must accept a cost intermediate between their minimum and maximum costs. It is clear, however, that any definition of "fair share" should imply very similar, if not identical, unit costs for D2 and D3. This condition is clearly not met by the solution in Table 2, even though this maintains minimal total costs. A similar situation holds for users D1 and D5, although costs are not identical. The solution in Table 2 does yield unit costs for D1 and D5 which do not differ greatly, but another minimal-cost algorithm might equally well have yielded unit costs of 1862 for D1 and 1973 for D5.

The flexibility introduced by the existence of 106 feasible arcs (rather than the minimum number of 39) permits a minimum-cost solution which in some way allocates a "fair share" of costs (a term still to be defined) to each user. This leads to consideration of the second goal of the Maize Board, i.e. the second level of optimization.

The second goal is thus to ensure a measure of fairness towards users. Both the definition of "fairness" and the achievement of fair solutions have to be discussed.

The method first obtains the minimal total rail cost, and then selects a solution which is the most "fair" or "equitable" while still satisfying the minimal-cost condition. It may turn out that some departure from minimum cost could result in a large improvement in fairness, yielding a solution which may be preferable. Experience with problems of real-life size shows, however, that there has always been sufficient flexibility within the number of feasible arcs to allow a solution to be obtained which was accepted as fair by the users, and for this reason as well as the difficulty of quantifying compromise criteria, justified the choice of method. The method is to first optimize according to the first criterion, maintain this level as a "goal" and then seek optimization of the second, which is a simple form of goal programming.[1] The second criterion is a set of criteria which are conveniently handled by setting further goals, for each user as an extension of the goal-programming approach.

The rest of the paper deals with the second optimization problem. i.e. that of achieving the most equitable solution.

NON-EQUIVALENT OPTIMAL SOLUTIONS

In this transportation problem there exists a high degree of degeneracy in the system as a result of linear relationships between costs to different users. The situation thus arises that the number of feasible arcs is much greater than the number needed and there is therefore no unique solution to the primary problem. In the sample problem there were 106 feasible arcs as opposed to the 39 required for a non-degenerate solution, which is indicative of the high degree of degeneracy in the problem. This degeneracy effect is more pronounced when minimum distance rather than minimum cost is the primary objective. This is to be expected, as the linear relationships between costs arise primarily from the corresponding relationships between distances, but are also moderated by the non-linear relationships between cost and distance.

The degeneracy implies that many minimal-cost solutions exist. For example, various combinations of some or all of the 106 feasible arcs in the sample problem could be used in different ways to yield feasible minimum-cost solutions. Attempts can thus be made to improve the achievement of the second goal while maintaining minimal cost.

The problem is now to identify and find the minimum total cost solution which is in some sense "fair" to all users in terms of the criteria discussed in the last section. This is a non-trivial problem as a result of the flexibility within the minimum-cost goal discussed in the preceding paragraph. Some acceptable definition of "fair" is needed in order to achieve this second goal.

The approach to defining "fair" which turned out to be most acceptable to the client (the Maize Board) was: Firstly, taking into consideration the known minimum total cost which has to be borne, and the maximum and minimum unit costs for each user, define in some way an ideal "equitable" unit cost, somewhere above the minimum unit cost for each user, such that total costs with these equitable unit costs approximate to the known (minimum) total cost (irrespective of actual feasibility of these ideal unit costs).

A number of possible definitions of these ideal equitable costs are discussed later in the paper. All these definitions are, however, based on the assumption that every user should accept a *pro rata* share of the unavoidable excess of minimum total cost over the total of minimum costs for each user. For example, in the sample problem in Table 1, Table 2 shows that:

$$\text{Minimum possible total cost} = 1.2166 \times 10^8$$
$$\text{Total of individual minimum costs} = 1.1831 \times 10^8.$$

The difference of 3.35×10^6, i.e. an average unit cost of 47, has to be shared among the users. Table 3 shows the equitable costs derived from the third definition of "equitable" discussed later.

Given such a set of equitable costs then, the goal of "fairness" is defined as minimization of the sum of deviations from total equitable cost (unit cost × quantity) over all users.

HEURISTIC METHOD

This method is based on a type of "chain" approach used in the primal–dual optimization method (cf. Simmonard[2]) but subject to the constraint of not increasing computational cost too severely. The following is a brief outline of the method.

TABLE 3. IDEAL EQUITABLE COSTS AND BEST SOLUTION FOUND

User point	Equitable unit costs	Unit costs in goal programming solution
D1	1915	1915
D2	1616	1568
D3	1616	1616
D4	1885	1908
D5	1923	1923
D6	1908	1908
D7	1950	1982
D8	1968	1968
D9	1531	1534
D10	379	373

Step 1

Start with some optimal solution, and lists of suppliers, users and feasible arcs. Select the first user in the list and go to Step 2.

Step 2

Find a pair of suppliers both connected to the given (or "current") user by feasible arcs, and not previously considered (for this user). If these suppliers have been found, go to Step 3; if none can be found, select a new current user from the list and repeat Step 2.

Step 3

If it is possible to shift the tarriff payable by the current user nearer the prescribed equitable value by shifting supplies between the two suppliers, then go to Step 4. If not, return to Step 2 to find a further pair of suppliers.

Step 4

Attempt to form a "chain" of feasible arcs starting and ending with the two user–supply arcs identified in Steps 2 and 3 (the same user, but two suppliers). The required shift of supply can be achieved by increasing flow in the first arc, reducing it in the second, and so on.

It is necessary to form the chain in such a way that these transfers will still result in a feasible solution (i.e. no negative flows). Furthermore, to prevent cycling, an arc is introduced only if the cost to the user node of this arc does not deviate further from the equitable tariff than before, although the direction of deviation may change. This is in contrast to the corresponding deviation for the current user, which is forced to reduce in magnitude without changing sign.

If such a chain is found, the maximum shift of suppliers through the chain, consistent with the above constraints, is implemented; then repeat Step 3 with the same pair of arcs. Otherwise return to Step 2.

This procedure is continued until no further improvement can be achieved.

There are two main disadvantages to this approach. First of all there is no guarantee that the closest approach to the equitable cost is obtained for each user. There may be a still better "equitable" solution. Secondly, although computational time is satisfactory if the number of feasible arcs is small, as this number increases the time rises rapidly (approximately as the cube of that number). Nevertheless for smaller problems, this heuristic method can be substantially more efficient computationally than the exact method described next.

GOAL PROGRAMMING METHOD

The secondary problem can also be formulated as a linear programming (L.P.) problem of the goal programming (G.P.) type.[1] In G.P., objectives or goals are incorporated into the constraints as "goal" equations. Each goal (in this case the prescribed equitable cost for each user) is represented by an equality constraint with the addition of two special deviational variables (akin to slack variables) which represent under- and over-achievements of the goals. It is the sum of these deviational variables that is minimized in the G.P. objective. In formulating the G.P., the same weights are given to under- and over-achievements and equal weights to all goals (i.e. to each user). The former may not be strictly necessary since the only disadvantage in over-achievement (apart from "political" disadvantages) is under-achievement elsewhere.

Formulation

Objective function:

$$\min \sum_{j=1}^{N} (u_j^- + u_j^+)$$

which minimizes the sum of absolute deviations from the equitable tariffs to each user.

The following constraints apply:

(i) goal constraints, which define the deviational variables in terms of the other variables:

$$\sum_{l \in V(j)} c_l y_l + u_j^- - u_j^+ = I_j, \quad j = 1, \ldots, N;$$

(ii) demand constraints (see note below):

$$\sum_{l \in V(j)} y_l = D_j, \quad j = 1, \ldots, N;$$

(iii) supply constraints (see note below):

$$\sum_{l \in L(i)} y_l = S_i, \quad i = 1, \ldots, M;$$

(iv) non-negativity constraints:

$$u_j^-, u_j^+ \geqslant 0, \quad j = 1, \ldots, N$$

and

$$y_l \geqslant 0, \quad l \in V(j) \text{ for all } j$$

where:

y_l = the quantity (in tons) of maize to be railed on feasible arc l;
c_l = corresponding rail tariff (or distance);
D_j = demand by user j;
S_i = supply from supplier i;
$V(j)$ = set of feasible arcs connecting user j;
$L(i)$ = set of feasible arcs connecting supplier i;
I_j = goal cost (corresponding to the prescribed equitable tariff) for user j;
u_j^- = under-achievement of the jth goal;
u_j^+ = over-achievement of the jth goal;
N = no. of users;
M = no. of suppliers.

These restrictions on $V(j)$ and $L(i)$, the sets of feasible arcs associated with user j and supplier i respectively, ensure that the total costs are minimal.

Note

The formulation assumes that total demand equals total supply. In practice, however, supply usually exceeds demand and thus a dummy demand point is introduced to take the excess supply at zero cost. For convenience in the L.P. formulation, the dummy demand is not included in the constraint (ii) nor explicitly in constraint (iii), but constraint (iii) must then, in this case, be replaced by the inequality

$$\sum_{l \in L(i)} y_l \leqslant S_i$$

for those suppliers for whom the arc to the dummy node is feasible (i.e. supply to the dummy node, or equivalently supply slack may be included in the optimal solution).

The feasible arcs to the dummy node are easily obtained from the solution to the primary problem, the standard transportation problem. Since costs on all arcs into the dummy node are by definition zero, the optimality condition becomes:

$$u_i = -v_d \text{ if the supplier-}i\text{-to-dummy arc is feasible,}$$

$$u_i < -v_d \text{ otherwise,}$$

where v_d is the dual variable corresponding to the dummy node and u_i is the dual variable corresponding to user i. Thus the inequality form of constraint (iii) applies to user i if

$$u_i = \max_{i'} u_{i'}.$$

Solution of the problem

The problem has been solved on a CDC Cyber 174 machine by use of the APEX linear programming package. A simple matrix generator in FORTRAN was written to convert the solution of the primary problem into the MPS format required as input to APEX. The standard output from APEX was stored on a special file, from where it could be retrieved by a report generator. The one disadvantage of this approach, especially when distances are to be minimized, is the problem of linearity of the objective function. For example, if there are two or more user points at the same location and the goals cannot be met, then the combined over- or under-achievement is usually allocated to one user only on an arbitrary basis by the simplex algorithm. This problem does not arise, however, if all the equitable costs can be achieved within the constraints, since then all under- and over-achievements are zero.

The solution of the sample problem obtained by the goal programming method is shown in the last column of Table 3.

The user unit costs in the optimal solution are not all equal to the ideal equitable costs because the ideal solution does not exist. Nevertheless, the solution shown in Table 3, is clearly an improvement on the initial solution in Table 2. For users D1, D5 and D6 the respective ideal goals are met exactly, whereas in the initial solution both the over- and under-achievement were greater than 20 in all three cases.

For users D2 and D3 there is also an improvement, although there is still an under-achievement of 48 for user D2. This demonstrates the one weakness of the approach, which is that for users at the same place, if the ideal cost cannot be achieved then all the discrepancy is arbitrarily allocated to one of these users. This inadequacy can usually be overcome by inspection, however.

EQUITABLE COSTS

A vital aspect of the secondary optimization problem is the definition of equitable unit costs which should not merely be fair, but also as far as possible practically realizable.

The following definitions of "equitable" were employed and compared.

Definition 1

For each user there is a minimum unit cost (or tariff) which he must bear in any optimal solution to the primary problem. This tariff is computed for each user independently of all the others, by considering the cheapest feasible arcs along which all demands can be met. These minimum tariffs are summed and subtracted from the optimal cost, previously determined, and the difference is divided by the total quantity of maize railed. This result multiplied by each user's demand is added to the minimum tariff for each user to give a value defining an equitable cost for each user. Clearly the sum of these costs gives the optimal total cost, but a particular user's equitable cost may not be realizable.

Definition 2

In this case, both the minimum and maximum tariffs within primary optimal solutions are computed for each user. As before, the minimum tariffs are summed, the sum is subtracted from the optimal cost, and the difference is divided by the total quantity of maize railed. This result is multiplied by each user's demand to give a value, A, say. The difference between the maximum and the minimum tariffs is also computed for each user, and is denoted by B. The minimum of A and B is added to the minimum tariff of each user which then gives the equitable costs for the user. These are now realizable individually but sum to an unrealizably low level.

Definition 3

Both the minimum and maximum tariffs for each user are obtained as before and the average of these minimum and maximum costs are computed. The averages are summed and subtracted from the optimal cost and the difference is divided by the total quantity of maize railed. This result, which may be negative, multiplied by each user's demand, is added to the average cost for the user. The "equitable" costs sum to the correct value and appear to have more chance than in Definition 1 of being actually realizable within an optimal solution. Furthermore Definition 3, besides seeming to be the most satisfying of the three as regards fairness to all users, also yielded the best practical results.

CONCLUSION AND IMPLEMENTATION

A comparison between the two approaches shows that in all instances the G.P. approach yields a more "equitable" solution.

As far as computational time and cost are concerned, the heuristic method is faster when the number of feasible arcs is comparatively small: for instance, a problem involving about 500 feasible arcs takes 264 seconds to run using the heuristic approach and 325 seconds with the G.P. method. This difference is, however, not very large in terms of computer cost, and for problems involving a large number of feasible arcs, the G.P. method is much faster and correspondingly far more economical. For example, a problem involving over 3000 feasible arcs ran for 1250 seconds using the heuristic approach and 410 seconds using the G.P. method.

The G.P. method has now been implemented, although most of the problems encountered are of average size. The cost per run is thus usually a little higher than with the heuristic approach, but this disadvantage is compensated by the better solutions obtained and by the security given against problems arising if a large number of feasible arcs occur (the number of which cannot be predicted from the size of the original transportation problem!).

The Maize Board now uses this program on a regular basis, for the two types of maize produced. The program is used to obtain preliminary indications, before harvesting, of the points from which each user will be supplied (on the basis of estimated harvests) as well as to obtain the final supply schedule. The data required by the model form part of the information system maintained by the Maize Board, and thus no additional data-gathering problems are involved in operating the model.

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