On Calculation of Radiation Field Integrals for Higher-Order Basis Functions in Conical Thin Wire MoM Formulation

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Abstract: - An efficient method of integration for far field calculations is derived. The method applies to the integrals arising from calculation of far field pattern with higher-order polynomial basis functions and moment method. The integral under consideration is a product of a power and exponential functions. Depending on the electrical length of the integration path and the required accuracy, either integration by parts or small parameter expansion is applied, in a recursive manner. The speed performance of a Matlab implementation indicates that the presented method is favorable compared to the commercial software "WIPL-D" used as a reference.

Key-Words: - antenna radiation patterns, moment methods, numerical analysis.

1 Introduction

There are cases when modeling of even simple antennas and scatterers may take a relatively long computing time, e.g. when optimizing the geometry with respect to radiation parameters. This often happens due to the need to calculate a radiation pattern or radar cross section (RCS) for many excitations or simply at a large number of points. Calculation of a radiation pattern typically involves a summation of partial electric fields produced by individual current components [1]. Each partial electric field is an integral of the Green's function and the respective current (basis function). It is easy to deduce that the computational complexity of this task scales as $O(N \cdot M)$, where N is the number of unknowns (directly related to basis functions) and M is the number of angular points, at which the radiation pattern is to be calculated.

The process of solving the system and obtaining unknown currents requires $O(N \cdot \log N)$ to $O(N^3)$ operations [2]. Thus, when the impedance matrix size is not large (say, under 1000 unknowns), most of the total time used for simulation is often consumed by integrations and summations involved in the far field estimation.

In the field of ultra wide bandwidth (UWB)/short pulse (SP) arrays, the synthesis techniques [3], [4], [5], often require computation of radiated fields over multiple radiators and over many steering directions.

Evaluation of the far field is often considered a trivial task. This is the case where (a) the basis function is a pulse or a piecewise linear function, or a piecewise sinusoidal basis function [1], and (b) the Green's function is a free space Green's function or may be expanded into a superposition of such [6]. The piecewise sinusoidal functions permit exact analytical integration. The general expressions for an integral of exponent and power function may be found for instance in [7]. For electrically very short segments, the expressions may be reduced to a central point rule [1]. For somewhat longer segments, as the integrand has no singularity, a simple numerical quadrature may be readily used as done by many authors [1], [8].

Traditionally, the integration for power basis functions, including those mentioned above, is performed either by a direct analytical integration or numerically [1], [8], [9]. However, such an approach is usually limited to a specific basis function and may, subject to accuracy and speed, be restricted in the electrical size of the wires. As it will be shown in this paper, the high powers of polynomial basis functions quickly give rise to large errors due to limited computer accuracy

In this paper, a different analytical approach has been applied for the integration involving higher-order power basis functions on thin wire segments. The resulting closed-form expressions, obtained by integration by parts, were found to suffer from a rapid loss of numerical accuracy for electrically short segments. This problem has been addressed by utilizing a series expansion at low frequencies. The obtained formulae were analyzed from the point of view of numerical accuracy. Further, the error estimating expressions for low and higher frequencies were combined to give the point of optimum to separate the regions of applicability of the series and integration by parts expressions, as well as to give an estimate for the maximum error due to the developed combined approach.

To enhance performance, the recursive properties found in the expressions have been utilized and the resulting algorithm implemented in Matlab [12]. Comparison with the output from a commercial program WIPL-D [10] has shown high efficiency of this algorithm.

2 Derivations

2.1. The Function Under Consideration

Computing the far field from the polynomial-approximated current on thin-wire antennas and scatterers [8], [10], [11] leads to integrals of the form,

$$F_i^u(\xi) = \int_{z_1}^{z_2} z^i e^{\xi z} dz \qquad (1)$$

where $\xi = j\beta \vec{u}_{r'}\vec{u}_z \cos\alpha$. This integral represents the normalized electrical field produced by a straight wire segment. The geometry of the wire segment is to be modeled by a truncated cone with opening angle α and axis matching $\boldsymbol{z}$ axis of a segment's local coordinate system. The factor z^i in the integrand is a term of a higher-order power/polynomial basis function. The exponent in the integrand represents the fast-changing factor of the free-space Green's function, expanded asymptotically, in the Fraunhofer region. The integration is done from the beginning z_1 of the cone to its end z_2, along the cone axis. The wire (cone axis') unit vector is denoted as $\vec{u}_z$, and the unit vector to the observation point is referred to as $\vec{u}_{r'}$. The other notations used are: i is the order of power basis function, and β is the propagation constant. To highlight the physical meaning, the product of the unit vectors can be expressed as cosine of the angle between them: $\vec{u}_{r'}\vec{u}_z = \cos\theta$. With this in mind, the function will also be referred to as $F_i^u(\theta(\xi))$.

2.2. Example of Numerical Instability

For a uniform current distribution (i=0), the integral (1) may be written as

$$F_0^u(\theta(\xi)) = \int_{z_1}^{z_2} e^{\xi z} dz = \left(e^{\xi z_2} - e^{\xi z_1}\right)/\xi$$

$$= (z_2 - z_1) e^{j\beta \frac{z_1+z_2}{2} \vec{u}_{r'}\vec{u}_z \cos\alpha} \frac{\sin\left(\beta \frac{z_2 - z_1}{2} \vec{u}_{r'}\vec{u}_z \cos\alpha\right)}{\beta \frac{z_2 - z_1}{2} \vec{u}_{r'}\vec{u}_z \cos\alpha}$$

Both closed forms for this integral (with exponents and sinuses) are numerically unstable (as $\frac{0}{0}$) at angle θ equal 90 degrees and also for conical wires degenerated into disks (when the length of cone is reduced to zero). The exponential form of the expression is also prone to errors due to subtraction of closely valued exponents. This happens when $\beta(z_2 - z_1)\vec{u}_{r'}\vec{u}_z \cos\alpha \approx 0$, for example (a) for electrically short wires, (b) for values of angle θ close to 90 degrees, and (c) for the cones that are close in shape to disks. The accuracy in the result is lost completely when the first, linear, term in the expansion of exponent ($e^{\xi z} \approx 1 + \xi z + \ldots,\ |\xi z| << 1$) becomes smaller than the uncertainty of the floating point number representation $\frac{\varepsilon_0}{2}$ used in computations. This may be written as the inequality: $|\xi z| \le \frac{\varepsilon_0}{2}$. The scenario where the accuracy is lost is equivalent to taking into consideration only the static field, i.e. $e^{\xi z} \approx 1$, and neglecting the radiating terms.

It is possible to estimate the error due to the limited accuracy of floating point number representation. The inequality may be re-written as $1 \le \frac{\varepsilon_0}{2|\xi z|}$. Considering the left- and right-hand sides equal, and taking the unity as a relative error of 1, the fractional inaccuracy due to subtraction of closely-valued terms δ_{F_0} may be evaluated as

$$\delta_{F_0} \le 2\frac{\varepsilon_0}{2}\frac{1}{|\xi z|}, \quad |\xi z| \ll 1 \qquad (2)$$

This expression quantifies the numerical instability due to subtraction of exponents with close powers.

2.3. Lower and Higher Frequency Formulae

In the situation described in the previous section, it is convenient to expand both exponents into a Maclaurin series with respect to the parameter ξ:

$$F_0^u(\xi) \cong (z_2 - z_1) + \tfrac{1}{2}\left(z_2^2 - z_1^2\right)\xi + \ldots, \quad |\xi| << 1.$$

Such procedure may also be repeated for higher-order power functions. This is however a somewhat laborious task.

A simpler method to calculate the integral when the parameter ξ is small, and for arbitrary $i \ge 0$, is to expand the exponent in the integrand into a Maclaurin series and integrate the resulting combined series analytically, as shown in (3).

$$\begin{aligned} F_i^u(\xi) &= \int_{z_1}^{z_2} z^i e^{\xi z} dz \\ &\cong \int_{z_1}^{z_2} z^i \left(1 + \xi z + \tfrac{1}{2!}(\xi z)^2 + \ldots\right) dz \\ &= \tfrac{1}{i+1}\left(z_2^{i+1} - z_1^{i+1}\right) + \tfrac{1}{1!}\tfrac{1}{i+2}\left(z_2^{i+2} - z_1^{i+2}\right)\xi \\ &\quad + \tfrac{1}{2!}\tfrac{1}{i+3}\left(z_2^{i+3} - z_1^{i+3}\right)\xi^2 + \ldots \end{aligned} \quad (3)$$

It is useful to note by writing the next, $(i+1)^{\text{th}}$, term

$$F_{i+1}^u(\xi) = \tfrac{1}{i+2}\left(z_2^{i+2} - z_1^{i+2}\right) + \tfrac{1}{1!}\tfrac{1}{i+3}\left(z_2^{i+3} - z_1^{i+3}\right)\xi + \ldots$$

that some parts of F_i^u may be reused in computing F_{i+1}^u. When computing several terms of a polynomial, the re-usage enables a partially recursive process and may therefore save some computational efforts.

At higher frequencies the electrical length of the integration interval may be too long to apply the expression (3). Then one may integrate (1) by parts. The resulting expressions shown in (4) may also be used in a recursive manner.

As shown later in the paper, the accuracy of results produced by the expansion (3) worsens for larger values of parameter ξ. On the other hand, the accuracy of formula (4) is improving with growth in ξ. Therefore, using (3) for small ξ and (5) for large ξ may be expected to give the best accuracy. The respective domains may then be separated at a break point. It is assumed that such a point exists. The coordinate of this break point, ξ_0, will be identified later in the paper.

Summarizing the above-mentioned, it is possible to write a procedure for calculating the value of the function $F_i^u(\xi)$ for all the powers of a polynomial over a specified wire segment as

$$F_i^u(\xi) = \frac{1}{\xi}\left(z^i e^{\xi z}\Big|_{z_1}^{z_2} - iF_{i-1}^u\right), \quad |\xi| > \xi_0 \quad (4)$$

$$\begin{aligned} F_i^u(\xi) &\cong \tfrac{1}{i+1}\left(z_2^{i+1} - z_1^{i+1}\right) + \tfrac{1}{1!}\tfrac{1}{i+2}\left(z_2^{i+2} - z_1^{i+2}\right)\xi \\ &\quad + \tfrac{1}{2!}\tfrac{1}{i+3}\left(z_2^{i+3} - z_1^{i+3}\right)\xi^2 + \ldots \\ &\quad + \tfrac{1}{(n-1)!}\tfrac{1}{i+n}\left(z_2^{i+n} - z_1^{i+n}\right)\xi^{n-1}, \\ &\qquad |\xi| \le \xi_0 \end{aligned} \quad (5)$$

These expressions allow partially recursive process of calculating the radiation integrals with hierarchical power basis functions.

Now it is left to evaluate the value of ξ_0, giving an optimum with respect to the best achievable accuracy. This requires error estimates for the series expansion, as well as for the exact expressions obtained by integration by parts.

2.4. Error Estimation for Lower Frequencies

The accuracy of the approximation (5) is defined by the number of terms left in the truncated series. The series is not slowly converging (convergence is better than one of sine or cosine functions) and this promises to give advantage in both speed and accuracy for the wire segments that have a short electrical length, as compared against the direct integration by parts. This feature is particularly important for curved structures geometrically approximated with a large number of small straight wire segments.

Considering a truncated series approximation (5) of order m of the function $F_i^u(\xi)$ in (1), the fractional accuracy of (5) may be evaluated by the ratio of its $(n+1)^{th}$ term to the first one:

$$\delta_{F_i}^{Sn} = \frac{\frac{1}{n!}\frac{1}{i+1+n}\left(z_2^{i+1+n} - z_1^{i+1+n}\right)\xi^n}{\frac{1}{i+1}\left(z_2^{i+1} - z_1^{i+1}\right)}.$$

To simplify further discussions, consider $z_2 = L, z_1 = -L$, where L is the half-length of the wire segment. The terms of the series (5) with even powers are zero, and only the odd-powered terms would be considered. The formula for estimating the fractional accuracy of expression (5) then reduces to

$$\delta_{F_i}^{SnL} = \tfrac{1}{n!}\tfrac{i+1}{i+1+n}\left|L\xi\right|^n \quad (6)$$

2.5. Error Estimation for Higher Frequencies

Derivation of an error estimate for the exact formula (4) is done in an iterative manner. For a zero-order power basis function, the absolute error in the exact expression for F_0^u may be evaluated as ε^1/ξ. Continuing with the other terms (e.g. $\varepsilon^1/\xi^2 + L\varepsilon^1/\xi$ for the next function, F_1^u), the absolute error for the function F_i^u with arbitrary index $i = 0,1,2,\ldots$ may be written as

$$\Delta F_i^u \le \tfrac{\varepsilon^1}{\xi^{i+1}} i!\left(1 + \xi L + \tfrac{(\xi L)^2}{2!} + \ldots + \tfrac{(\xi L)^i}{i!}\right) \cong \tfrac{\varepsilon^1}{\xi^{i+1}} i! e^{\xi L}.$$

The value of ε^1 may be found from the previously found estimate for the inaccuracy of F_0^u:

$$\frac{\varepsilon^1}{\xi F_0^u} = 2\frac{\varepsilon_0}{2}\frac{1}{\xi L},$$

so that $\varepsilon^1 = 2\frac{\varepsilon_0}{2}\frac{1}{\xi L}\xi F_0^u \le 2\frac{\varepsilon_0}{2}\frac{1}{\xi L}\xi 2L = 4\frac{\varepsilon_0}{2}$. Then the fractional inaccuracy for the terms with odd i may be estimated as

$$\frac{\Delta F_i^u}{F_i^u} \cong \frac{\frac{\varepsilon^1}{\xi^{i+1}} i!e^{\xi L}}{\frac{1}{i+1}\left(L^{i+1} - \left(-L\right)^{i+1}\right)} = 2\frac{\varepsilon_0}{2}e^{\xi L}\frac{\left(i+1\right)!}{\left(\xi L\right)^{i+1}}, \quad (7)$$

$$i = 1,3,5,\ldots, \quad \left|\xi L\right| \to 0$$

From this expression it is easy to see that the errors may accumulate very quickly, as the degree of polynomial (i) increases. This is very different from the behavior of the truncated series approximation, which changes much slower as there is no factorial dependency on the degree of polynomial.

It may also be noted that it is possible to derive the expressions for errors due to the non-recursive closed analytical form of the expression (1). It has the same form as the expression (7). As such, it possesses the same poor properties when applied to integration of high order polynomial basis functions on short intervals.

The both error functions, (6) and (7), are depicted in Fig. 1. The change in power i from 0 to 6 does not change the error of the truncated series approximation (6) much, at the same time producing dramatic effect in the error function (7) resultant from the integration by parts (4). From the point of view of the best accuracy that should be possible to achieve by combining the functions (4) and (5), the figure illustrates existence of an optimum value of ξ_0 for each pair of i and n. Considering an example with i=0 and n=1, a better accuracy may be obtained via the truncated series until approximately $\xi_0 = 2.1\cdot 10^{-8}$. After that point either a higher degree truncated series i, or the exact expression (4) should be used.

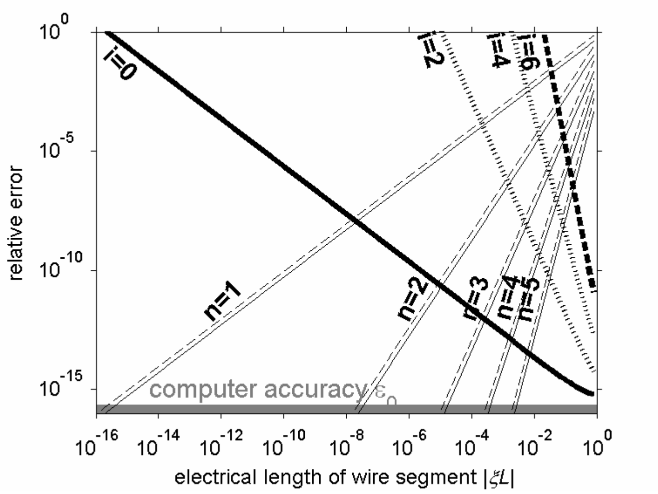

Figure 1. Relative inaccuracies in calculating integrated-by-parts functions and their truncated series approximations versus the electrical length of wire segment. Thick lines correspond to the function $F_i^u(\xi)$ with value of power *i*=0,2,4 and 6. Thin lines correspond to the truncated Maclaurin series approximation of degree *n*=1,2,3,4 and 5. These lines are either solid (*i*=0) or dashed (*i*=6). A gray strip at the bottom of the graph denotes the computer uncertainty level for double precision (i.e. $\varepsilon_0=2^{-53}\approx 1.1\cdot 10^{-16}$).

2.6. Finding a Break Point

Setting the derived error estimates for the series (6) and integration by parts (7) equal to one another gives an equation (where the exponent is approximated with its first term only, unity):

$$\frac{1}{n!}\frac{i+1}{i+1+n}\left(L\xi\right)^n = 2\frac{\varepsilon_0}{2}\left(i+1\right)!\left(\xi L\right)^{-i-1}.$$

This leads to an estimate for an optimum position to separate the domains for applying either the exact formula or truncated series approximation, shown in the expression (8):

$$\left|\xi_0\right| = \frac{1}{L}\left(2\frac{\varepsilon_0}{2}n!\left(i+1\right)!\left(1+\frac{n}{i+1}\right)\right)^{\frac{1}{(n+1+i)}} \quad (8)$$

This expression is numerically expensive. It is possible to reduce the computational costs by pre-tabulating the function of two arguments, n and i. With a careful selection of tabulated values, this approach can still lead to nearly optimal in terms of accuracy results.

3 A Note On Computing the Far Field Series

The total electrical field at a point is formally calculated as a sum of fields produced by each wire segment. The field due to a single wire segment is a sum of components due to hierarchical basis functions. To compute this double series more accurately, it is better to interchange the order of summation, so that the field components corresponding to the same power i for all of the wire segments would be added up in the inner loop:

$$\sum_{wires}\sum_{i} \Rightarrow \sum_{i}\sum_{wires}$$

It may be noted, though, that rearranging the summation this way may lead to a drop in speed of summation cased by a lower efficiency of usage of cache present in most of the currently available computer systems. This note however is valid only where the amount information related to the geometry of wire requires an amount of memory that is substantially larger than the size of cache.

4 Results

Typically, the achievable accuracy for radiation pattern measurements done in an anechoic chamber is around 40dB, which corresponds to the relative inaccuracy in the electrical field magnitude of 10^{-2}. From the plots in Fig. 1, it is easy to see that the exact function (4) resulting from integration by parts is incapable of this accurate reproduction of the far field for short wire segments when high-order polynomials are used to characterize the current. For this situation, an adaptive algorithm for computing the functions (4) and (5) has been developed, implemented in Matlab [12], and tested with Maple [14].

The testing was done by comparing the numerical calculations produced within Matlab against the results of closed-form analytical calculations performed in Maple. Matlab uses double precision number representation that gives up to 16 significant figures. Maple was set up to provide reference results with over 100 significant digits. The testing has confirmed accuracy of the combined approach.

For a low number of wire segments and high number of points in far field, the performance of the calculations done within Matlab [12] (without any compilation) was found to be only 20% slower than the speed of the commercially available WIPL-D. This confirms the efficiency of the proposed algorithm. Taking into consideration (a) the speed improvement potential from code optimization and compilation [13], (b) the fact that program WIPL-D [10] does all calculations with single precision, and (c) that Matlab uses a slower double precision, it is expected that the efficiency of the proposed algorithm is superior to WIPL-D's.

As a final remark, it may be noted that the best achievable accuracy decays as the degree of polynomial is increased. At the break point, the accuracy of results cannot be improved using the present approach. However, for practically usable (up to about 9) powers referred to [11], the available accuracy still exceeds the measurement errors by a substantial margin.

5 Conclusion

The expressions for estimating far field due to higher-order polynomial basis functions within the thin wire approximation in the method of moments have been developed.

The calculations have been branched. At lower frequencies, a Maclaurin series expansion has been applied. At higher frequencies, a recursive expression based on the integration by parts has been utilized. Formulas for evaluating errors in the resulting far field estimates due to numerical round-off effects have been developed for both higher and lower frequency branches. An accuracy-optimum boundary separating the use of higher- and lower-frequency formulae has also been found.

An algorithm utilizing the recursive properties of the two approaches has been developed and implemented in Matlab. The accuracy of the results has been validated against analytical computations done in Maple with virtually unlimited accuracy.

A comparison of Matlab code against commercial electromagnetic modeling software WIPL-D has demonstrated the high performance and good potential for the proposed approach.

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