Some Examples of Non-linear Systems and Characteristics of their Solutions

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Abstract

Complexity science is often seen as the science of emerging non-linear phenomena. In this paper we discuss some emerging aspects of non-linear solutions in physics. These solutions owe their elegance and simplicity to the complex non-linear structure of the equations, a structure which is dictated by the symmetries of physics. A central theme in these non-linear solutions is that the magnitude of the driving term (or the initial cause in more mundane language), is of little influence on the final solution. In linear approaches one would normally exploit the smallness of the source term by constructing solutions order by order. The non-linear solutions have a very different nature and cannot be constructed by such perturbative means. In contrast to certain other applications in complexity theory, these non-linear solutions are characterized by great stability. To go beyond the dominant non-perturbative solution one has to consider the source term as well. The parameter freedom in these equations can often be reduced by self-consistency requirements. We attempt to assess a possible role of this type of solutions in general complexity theory. In particular we stress the possibility that the complexity of the equations is beneficial rather than detrimental towards the solution of these non-linear equations, as long as this complexity reflects fundamental aspects or principles in the description of the system.

Keywords: Complexity Theory; Non-linear equations; Emerging Solutions

Introduction

The fundamental theories of physics, such as Quantum Electrodynamics (QED), Quantum Chromodynamics (QCD) and General Relativity are all associated with fundamental dynamical equations, which are called equations of motion. These are differential equations (usually non-linear) for the relevant space-time functions. In most applications of these theories (in particular in scattering problems) one calculates physical expressions order by order to avoid the complexities of non-linearities. Advanced techniques and software can be used to calculate the higher order terms, which become more complex the higher order they are. However, even for very small driving terms there is no guarantee that the resulting series converges. One can also attempt to solve these equations of motion non-perturbatively by dealing first with the non-linearities, thereby sidestepping the divergence problems. Simplified non-linear equations may admit exact solutions, while the details can be handled in subsequent iterations. Now the complex structure of the fields and coupled equations makes such an exercise problematic in physics. However, we have found that the very nature of these complex equations and entities permits the construction of elegant and stable solutions in certain fundamental problems (Greben, unpublished). Apparently, nature has "chosen" these complex structures for good reason as they lead to some very special desirable properties.

Clearly, we cannot expect that the system equations of more mundane problems in complexity theory are grounded in principles and structures of similar depth. Nonetheless, the fact that the elegant basic solutions (modes) would not emerge in these physics examples if non-linear terms had been dropped may contain an important message. It suggests that the (non-linear) complexity of system equations should be respected and that non-linearities can often be beneficial in constructing stable and dominant solutions. Hence, rather than avoiding and/or approximating non-linear terms, it may be more appropriate to treat the non-linearities in a rigorous way. One may even want to introduce additional non-linear terms to permit the construction of specific non-linear modes. Just like nature has selected specific mechanisms to enable a feasible and (self)-consistent reality, ecological, social and/or economic mechanisms may have evolved which survived the instabilities of historical developments. It might well be that these mechanisms can also be described by appropriate non-linear feedback loops.

In describing an ecological, social or economic system one might follow the accepted domain expertise and introduce all the relevant terms with adjustable parameters. However, we can also take the lead from nature by studying how it has realized the consistent formulation of natural phenomena. It is quite possible that these natural mechanisms can also be applied in other contexts. That is why we feel that it is useful to study some of the equations of motion from physics, albeit in simplified form, and consider their possible application in complexity theory. This may also alert us to certain non-linear aspects of system equations which carry a more profound meaning then superficially expected. Such aspects could for example be responsible for the stability of seasons in a weather system that is generally considered chaotic.

In this paper we will discuss the properties of such emerging non-linear solutions in physics with an eye on their relevance to complexity theory. We use the word emerging here to indicate that the non-linearities permit additional solutions which would not be present in the absence of such non-linearities and are distinct form perturbative solutions which correspond to the linearized equations. Because of the complexity of the physics solutions we will not discuss the technical properties of these solutions, but rather study some mathematical properties of the simplified equations.

Non-Linear Solutions in Physics

The fundamental theories in physics are based on deep principles, such as gauge invariance in Quantum Electrodynamics (QED) and Quantum Chromodynamics (QCD), and covariance in general relativity. All of these theories lead to fundamental dynamical equations, which are called equations of motion. While these equations are highly non-linear and complex in nature, they still hide the original symmetries that are impose $\nabla^2 A = \quad gf^2 + g\nabla A^2 + g^2 A^3$ d originally, and therefore unexpected elegant solutions with extraordinary properties might well emerge. Now it is common to treat the scattering problems in physics with perturbative methods, i.e. in terms of expansions in the weak coupling constants of the field theory. These methods avoid a full treatment of the non-linear aspects by introducing terms order by order, raising questions of ultimate convergence and self-consistency. Similarly, in general relativity it is common to describe curved space-time in terms of the weak modification due to the presence of local massive objects. Despite their successes, these perturbative theories only can describe certain aspects of nature. One might expect that non-perturbative solutions will describe other aspects and also carry properties distinct from the perturbative solutions. We will discuss the properties of some solutions of differential equations inspired by the equations of motion in QCD.

The equations of motion of QCD are non-linear equations for quantum fields. These fields are not ordinary numerical quantities having a particular value in each point in space-time. Rather they are non-commutative operators that turn a given state vector into another one. They also carry numerous indices, which makes their treatment even more difficult. Naturally, this is not the place to discuss these aspects, however, it is important to realize that all these aspects are probably essential for the correct description of our universe, as it would be unlikely that nature would have selected these complicated mechanisms without good reason. For the same reason we can expect that the solutions that emerge from these equations have unique properties which are essential for the way nature manifests itself. In this paper we present examples of such solutions, albeit in simplified form. The relevance of these solutions can then be investigated in the context of more mundane applications in complexity theory. For this reason we start with a symbolic representation of the equations of motion of QCD.

There are two distinct unknown functions in these equations, one representing the interaction (the gluon field A) and one representing matter (the quark field f). The mathematical structure then looks like:

$$\nabla^2 A = \quad gf^2 + g\nabla A^2 + g^2 A^3 \tag{1}$$

and

$$\nabla f = gAf \tag{2}$$

where g is a coupling constant, which usually functions as an expansion parameter and is considered small. Trivial solutions, where f is identical to zero, are not of interest. We can arbitrarily complicate these equations by introducing multiple A and f functions, but in the current context we want to demonstrate the consequences of the non-linearities. Naturally, a more complex structure contains a larger and more interesting scope of possible solutions, but without specific hints from nature such extensions might complicate, rather than elucidate, the power of system equations.

The symbol ∇ symbolizes an operator involving first order derivatives. In physics this could be derivatives with respect to time or space. In complexity theory most applications would involve time and not space, leading to a considerable simplification. However, a complication in system descriptions in complexity theory, which is not present in physics, may be the lag in response time, as most responses in real-world situations will not be instantaneous.

Let us now see whether the simplified equations which we consider, also display distinct elegant properties. First we set:

$$A = \frac{Z}{g}\ , \tag{3}$$

so that:

$$\nabla^2 Z = g^2 f^2 + \nabla Z^2 + Z^3, \tag{4}$$

and

$$\nabla f = Zf\ . \tag{5}$$

Hence, by a simple redefinition we have removed the dependence on the coupling constant from all terms except for the driving term f^2 .

Perturbative solutions are obtained by expansions in the coupling parameter g. Initially the non-linear terms in Eq.(4) are deleted and Eqs.(4-5) is solved iteratively. To be explicit, one writes:

$$Z = g^2 Z_1 + g^4 Z_2 + \cdots \tag{6}$$

and

$$f = f_0 + g^2 f_1 + g^4 f_2 + \cdots \tag{7}$$

and solves the equations order by order. For example,

$$\nabla f_0 = 0 \tag{8}$$

and Z_1 is a solution of the equation:

$$\nabla^2 Z_1 = \quad f_0^2 \tag{9}$$

If g is small we can terminate the series (6) and (7) after a few terms. As Z = O(g^2), the original function A in Eq.(3) is well-defined for g = 0 (in fact it is zero).

The non-perturbative solution of these equations proceeds very differently. First we construct a homogeneous solution of Eq.(4) by omitting the driving term $g^2 f^2$. Replacing the general derivative by a simple time derivative we obtain the following solutions:

$$Z = \frac{B}{t+\beta} \tag{10}$$

where B can assume three values: $B = 0 \text{ or } B = 1 \pm \sqrt{3}$ and the constant β is arbitrary and must be fixed by the boundary conditions. It reflects the translational invariance of the equation. The solution B = 0 corresponds to the perturbative solution and can be considered as the lowest order term in this solution. The two other solutions are genuine non-perturbative solutions, which “emerge” because of the presence of non-linear terms.

Given these three basic modes, we can now derive the form of the driving term:

$$f = \gamma(t+\beta)^B \tag{11}$$

where B assumes the three values indicated before. This solution contains another adjustable parameter γ, which is different from zero for non-trivial solutions. We can now iterate the solutions further in terms of g in order to study the finer details. Whether this series converges is not so important, as one can always introduce higher order terms in Eq.(1), to guarantee convergence. However, if convergence is not guaranteed, only the first few high order terms should be included. For divergent series the first few perturbative terms may still make sense as an asymptotic expansion, however, it would be meaningless to go to higher and higher order. For the non-perturbative solutions (B ≠ 0) A is singular for g = 0, as it is proportional to 1/g. Obviously, the non-perturbative solutions have a very different character than the perturbative one, where A is finite for g = 0.

There is also an exact solution for B = - 3/2, with

$$f^2 = -\frac{39}{8g^2}(t+\beta)^{2B} \tag{12}$$

In this case the normalization of f is fixed by the equations. Although the sign of this solution indicates that it might be unfeasible in this case, for other coefficients in Eq.(1), this solution may well be feasible. Both A and f are now singular for g = 0, so that this is a truly non-perturbative solution. Since there is in general no simple way to construct such incidental exact solutions, one might easily miss it in the solution of the non-linear equations. It is a testament to the richness of these non-linear equations that such incidental solutions exist. The reason we call this solution incidental is that it is not found through the standard construction of modes. In this case there exist other methods to uncover the solution Eq.(12) and therefore one may object to the characterization of such a solution as incidental. However, what we want to stress is that non-linear equations may have unexpected solutions, which just might have added importance because of their unique, and therefore stable, character.

Let us now discuss some general properties of these solutions. First of all, the three basic modes which have been found are very distinct. Hence, we can attribute great stability to them. However, a given initial value for f does not fix the whole solution, since f in Eq.(11) displays two free parameters. Other constraints will determine which mode will be selected. The driving term (the causal mechanism) therefore has an unpredictable influence on the nature of the full solution. This can be seen as an example of the butterfly effect: although the value of f has an effect on the future solution, its effect depends strongly on other aspects of the system. Secondly, we can wonder how transitions between different modes are possible in view of their stability. Such transitions are possible if we generalize Eq.(1) by multiplying each term with suitable coefficients. In practice these coefficients would also vary with time (hopefully slowly), so that at different times, different modes would apply. It might now happen that for certain values of these parameters two non-perturbative modes coincide. At such instances one can easily visualize a transition between two basic modes. It would be of interest to identify such bifurcation (?) points, as they might signal the transition from one regime (e.g. a growth scenario) to another (e.g. a reduction scenario). For a simple set of equations with constant coefficients, the value of the critical parameters would be easily determined.

Are there any properties which manifested themselves in our QCD calculations (Greben, unpublished) which are either hidden or absent here? One property we find in the QCD case is that the non-perturbative solution shares certain properties with the perturbative solution, but at the same time is much more stable. This does not appear to be the case here, as the solutions in Eq.(11) are all very distinct for different values of B. The *stability* of these solutions distinguishes them from other phenomena in complexity theory where the *instability* of the critical solutions is emphasized (Bak and Chan, 1991). Another aspect of the QCD solutions may be more relevant to applications in complexity theory. Although the source terms play a minor role in the fixation of the modes, they play an essential role in the formulation of the equations, as it is their structure and properties which fix the form of the equations. Carrying this idea to complexity theory, it would be essential to have sufficient (domain) knowledge of the quantities described in the system to derive the appropriate equations and couplings, even though the various causes may play a minor role in fixing the global modes. The form of the respective quantities in the systems equations can be formulated in terms of Ansatze, which subsequently can be tested for consistency. Self-consistency is therefore an important tool in determining the correctness of the equations and the solutions. A third aspect, which we encountered in the QCD case, is that despite their relative unimportance, the driving terms plays an important role in fixing the details of the solution. The free parameters in the driving term can be adjusted to provide an optimal fit to the relevant data Subsequently, the quality of the fit can be assessed. (Self-) consistency can also play a role in fixing the relevant parameters. For example, in Eq.(12) we found that the simple parameters in Eq.(1) do not yield a feasible accidental solution.

The Relevance of Non-linear Solutions in Complexity Theory

Having looked at specific solutions for our simplified differential equations, and having quoted some results from physics, we now want to formulate our conclusions in more qualitative language so as to emphasize their possible role in complexity theory:

1. The coupled equations in the form (1-2), generalized to feature additional coefficients, may well be simple enough to function as a representative for some non-linear systems in complexity theory. The derivatives in systems theory would normally be limited to time derivatives, thereby simplifying the solution techniques. The solutions should be expanded around the main modes of the non-linear equations. One should also consider the possibility of incidental solutions.

2. The elegant solutions of the full QCD equations only emerge if the complexities of the equation are fully respected. For systems equations describing more mundane processes we also would expect that the existence of stable solutions is dependent on the presence of non-linear coupling terms and feedback mechanisms. Simplification or non-inclusion of crucial non-linear mechanisms may therefore eliminate the solutions of relevance.

3. Can we identify certain feedback mechanisms which are responsible for stable systems? Since stable systems are likely to be more representative of our environment than unstable ones, if only because of evolutionary pressures, we should investigate which specific non-linear constructions are responsible for stability in our everyday world. Physics can provide useful examples for these mechanisms. Although we cannot expect that social situations contain the same deep principles as physics, we might expect that some of the same mechanisms are present, helped along by social or cultural evolution.

4. Often, non-linear equations for real-world situations do not have simple solutions. Mathematicians or modellers would then rephrase or simplify the equations until they can construct a solution. However, this process may not lead to an elegant solution or a solution with amazing new (emerging?) properties or features. On the other hand, nature has often found amazing ways to cope with certain problems, leading to unique and desirable properties. Hence, we might want to copy such mechanisms within the context of system equations occurring in complexity theory.

5. Another amazing feature of nature is its hierarchical built up (Chaisson, 2001). System equations might also be constructed in a hierarchical way, whereby the main modes satisfy one set of equations, while the details are described at a different level of the hierarchy. Again, analogies to solutions found by nature could be beneficial to complexity theory. In some cases this might help us to select which terms to keep and which terms to drop in the systems equations at different levels in the hierarchy.

6. Nature often chooses special solutions: rather than selecting one of the general solutions it might chose the one exception to the general rule. Our incidental solution in Eq.(12) is a possible example. That is why it is easy to miss nature's solutions, as they often seem to violate a rule which is thought to be generally applicable. In physics these rules are known as no-go theorems, and more often then not they are being violated by nature. Spontaneous symmetry breaking is one of the modern phenomena where general symmetries are being violated. In the context of applications in complexity theory this might mean that one should carry out analytical rather than numerical studies, as the special solutions are hard to spot numerically.

7. The "agents"" in our systems analysis should be represented in the right way. A simplified representation of these agents (for example missing certain dynamical or possibly adaptive features) could spoil the potential solutions of our system. The domain knowledge about the driving terms (causal agents) is instrumental for the formulation of these properties.

8. In physics transitions from one equilibrium situation to another (or from one mode to another) are given by scattering matrices. Similarly in complexity theory, one could think of transitions from one main mode to another by means of a separate description using transition matrices. In physics, the transitional processes and the modal degrees of freedom require totally different treatment methods, the former often requiring more simple perturbative approaches. Further research is required to determine whether a similar situation applies in applications of complexity theory.

9. A more natural way to effect transitions in system equations is by allowing the coefficients in the equations to vary with time. When a point is reached that two modes coincide, a transition between the two is possible. Hence, the system equations should be analyzed in order to determine the conditions for these phase transitions or bifurcation points.

Summary

Often complexity theory deals with strong fluctuations in the solutions originating from a single - nearly unique - initial state. The solutions can converge to various different states, often called attractors. However, which attractor will be selected is often hard to predict. Using examples from physics, we also emphasize the stability of non-linear solutions and the irrelevance of the initial cause in determining the nature of the main modes. We notice that the transition between different modes in physics requires a separate description that is distinct from the description of the overall system modes. We suggest that such a description might also be called for in complexity theory, and recommend further study of this particular aspect. In physics, the elegance and stability of the mode solutions is only guaranteed if the full complexity of the equations is respected. We submit that a similar situation might apply in more qualitative applications of non-linear equations in complexity theory. Hence, the complexity of the system equations might be a virtue, rather than a problem. In particular one might want to introduce terms that increase the stability of the system, even if they make the equations look more complex. It is likely that evolution has selected such mechanisms over less stable alternatives.

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