A Novel Approach to Solving the Generalized Inverse Frobenius-Perron Problem

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***Abstract*—A new approach to solving a more general formulation of the inverse Frobenius-Perron problem, which requires the construction of a one-dimensional ergodic map with prescribed invariant probability density function and power spectral density, is presented. The proposed approach relies on a novel technique for generating distinct maps with the same invariant density, and which facilitates selection of the structural characteristics of each map in advance. We consider a new class of maps constructed with this technique, the piecewise monotonic hat maps, and present an algorithm for selecting the map parameters to achieve simultaneous and independent prescription of the invariant density and multimodal power spectrum characteristics. This approach to solving the generalized inverse Frobenius-Perron problem is demonstrated by constructing several ergodic maps with the beta invariant density as well as unimodal and bimodal power spectra with distinct mode center frequencies and bandwidths. We conclude that the proposed approach provides a means for generating more realistic models of systems and processes as compared to existing methods.**

I. Introduction

The problem of constructing a dynamical system model with prescribed statistical properties, which is colloquially referred to as the inverse Frobenius-Perron problem (IFPP) [1], is widely encountered in fields as diverse as physics, biology, economics and engineering. Solutions to the IFPP have been used to model phenomena such as the motion of fluids [2] and neural processes pertaining to the human olfactory system [3], and have led to the design of more robust rotary drill heads [4], more efficient and flexible digital radio-frequency memory systems [5] and improved waveforms for radar systems [6].

The conventional formulation of the IFPP requires the construction of an ergodic map, which corresponds to the evolution rule of a one-dimensional dynamical system model, such that the map has a prescribed invariant probability density function (PDF) [1]. Several solutions to this problem have been proposed [7]–[11]; however, with modern simulation environments striving to provide more realistic system and process models, there is growing interest in solving a more general formulation of the IFPP, where *both* the invariant density *and* the power spectrum of map trajectories are prescribed. The synthesis of structurally distinct maps sharing the same invariant density is also valuable from a theoretical perspective, as these maps reveal how certain statistical characteristics may emerge in systems. Hence, the development of analytical solutions to the IFPP which accommodate the simultaneous and general prescription of both an invariant density function and power spectral density is of relevance.

In this paper, we present a novel approach to solving the general formulation of the IFPP. We present a technique for generating distinct maps which share the same prescribed invariant density, and where the structural characteristics of each map (namely, the number of branches, branch support and branch completeness) are selected in advance. The technique uses monotonic random variable transformations, and accommodates a large class of invariant density functions; furthermore, the freedom to select the map structure provides the capability to manipulate the power spectrum of the map while the invariant density remains fixed. We investigate a new class of maps, referred to as *piecewise monotonic hat maps*, that are generated with the technique. The power spectra of these maps correspond to a superposition of spectral modes, where the structural parameters of the map accommodate the selection of each spectral mode's characteristics – i.e., its center frequency and bandwidth. We present an algorithm for selecting map parameters to simultaneously and independently prescribe the invariant density as well as these spectral characteristics. The proposed approach is demonstrated by constructing several ergodic maps which share the beta invariant density and possess distinct unimodal and bimodal power spectra, which are verified by numerical experiments.

The remainder of this paper is set out as follows. In section II, an overview of relevant literature on existing solutions to the IFPP is presented. The necessary mathematical background is presented in section III, whereas the technique for generating distinct maps with the same invariant density is presented in section IV. The class of piecewise monotonic hat maps, and the algorithm for selecting map parameters to realize a prescribed invariant density function and spectral characteristics, are presented in section V. Numerical experiments involving the generation of hat maps are presented in section VI, and the paper is concluded in section VII.

II. Overview of Literature

Baranovsky [7] proposed a solution to the more general formulation of the IFPP for shifted piecewise linear and complete maps (*complete* refers to each branch spanning the entire range of the map, in contrast to incomplete maps). Whereas the solution provides the flexibility to specify the invariant density (via topological conjugation), the power spectrum is restricted to being unimodal. The analysis indicates that this spectral limitation is characteristic of the maps considered, which limits their usefulness in solving the general IFPP.

Friedman [8] investigated the piecewise linear and incomplete Markov maps, and proposed a graph-theoretic approach to solving the conventional IFPP for these maps. In related work, Gora [9] proposed a solution to the IFPP for the class of piecewise linear semi-Markov maps; in this solution, the Frobenius-Perron (FP) operator of the map is represented as a matrix, referred to as the FP matrix. Both solutions [8], [9] are restricted to piecewise constant invariant densities, which is a consequence of the piecewise linear nature of the maps. Neither of these solutions addresses the power spectrum.

Isabelle [12] characterized the power spectrum of piecewise linear and incomplete Markov maps, thereby revealing its relationship with the eigenvalues of the $N \times N$ FP matrix. The power spectrum is a superposition of $2N$ spectral modes, where N of the spectral modes each corresponds to one of the eigenvalues; the argument of the eigenvalue is equal to the mode center frequency, whereas the eigenvalue magnitude is inversely proportional to the mode bandwidth. McDonald [10] generalized the solution of Gora [9] to achieve a bimodal power spectrum, where the mode bandwidths, but not arbitrary mode center frequencies, are selectable. The proposed technique involves the synthesis of a doubly stochastic FP matrix with prescribed real eigenvalues, and the subsequent construction of the map from the FP matrix. This approach was later generalized in [11] to the case of 3×3 FP matrices with prescribed complex eigenvalues, thereby providing freedom to select the characteristics of two of the spectral modes. However, both these solutions are restricted to piecewise constant invariant densities, and neither provide full control over the characteristics of all the spectral modes.

III. Mathematical Background

Let $\Omega \triangleq [0, 1]$. Consider a random variable X which induces the probability space $\Phi_X = (\Omega, \mathcal{B}, P_X)$, where $\mathcal{B}$ denotes the Borel σ–algebra on Ω, and where P_X is a probability measure. We consider probability measures that satisfy $P_X(B) = 0 \Leftrightarrow \mu(B) = 0$, where μ denotes the Borel measure and $B \in \mathcal{B}$; this implies that P_X is absolutely continuous with respect to μ, and that the distribution function F_X associated with the measure P_X is strictly increasing over Ω. Absolute continuity implies the existence of a PDF $f_X \in L^1(\Omega)$, where $||f_X||_1 = 1$, such that the density function satisfies $P_X(B) = \int_B f_X(x)d\mu(x)$ and $f_X > 0$ almost everywhere.

The proposed technique for constructing maps uses random variable transformations. In what follows, a class of surjective and piecewise monotonic Borel maps, which are used to transform the random variable X, is defined. A surjective Borel map $S : \Omega \to \Omega$ is piecewise monotonic [13] if there exists a partition $0 = x_0 < x_1 < \ldots < x_N = 1$ of the unit interval such that, for each $n = 1, 2, \ldots, N$ (finite N), (i) the branch function $S_n \triangleq S|_{(x_{n-1},x_n)}$ is a C^r function for some $r \geq 1$, and extendable to a C^r function on $[x_{n-1}, x_n]$, and (ii) $|S'_n(x)| > 0$. Using Villani's lemma [14], it can be shown that any such map is nonsingular; i.e., $\mu(B) = 0 \Rightarrow \mu(S^{-1}(B)) = 0$. An example of a map from this class is plotted in Fig. 1.

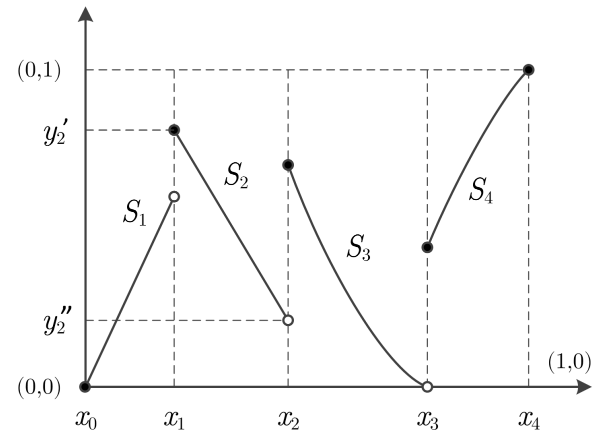

Fig. 1. Example of a piecewise monotonic map S.

Now consider the function $Y = S \circ X$. Since S is a Borel function, Y is measurable on $\mathcal{B}$; hence, Y is a random variable on the induced space $\Phi_Y = (\Omega, \mathcal{B}, P_Y)$, where P_Y is a probability measure. Also, since S is nonsingular, P_Y can be associated with a PDF f_Y. Let $N(y)$ be defined as $N(y) \triangleq \{n : y \in S((x_{n-1}, x_n))\}$. The two density functions f_X and f_Y are related by the expression (see, e.g., [13], [15])

$$f_Y(y) = \sum_{n \in N(y)} \frac{f_X \circ S_n^{-1}(y)}{|S'_n \circ S_n^{-1}(y)|}, \tag{1}$$

for almost all $y \in \Omega$. Since $f_X > 0$ almost everywhere, it is deduced from (1) that the distribution function $F_Y(y)$ is a strictly increasing and continuous map from Ω onto itself.

Whereas (1) is well-known, an analogous expression that relates the distribution functions F_X and F_Y may be derived. Let $y'_m \triangleq \sup\{S((x_{m-1}, x_m))\}$, and $M(y) \triangleq \{m : y \geq y'_m\}$. The expression relating F_Y and F_X is given by

$$F_Y(y) = \sum_{n \in N(y)} A_n(y) + \sum_{m \in M(y)} B_m, \tag{2}$$

where

$$A_n(y) \triangleq (-1)^{d_n}[F_X \circ S_n^{-1}(y) - F_X(x_{n-1+d_n})], \tag{3}$$

$$B_m \triangleq F_X(x_m) - F_X(x_{m-1}), \tag{4}$$

and where $d_n \triangleq 0$ if $S'_n(x) > 0$, and $d_n \triangleq 1$ otherwise.

This section is concluded by presenting a result regarding the existence of strictly monotonic transformations between random variables that induce the probability spaces Φ_X and Φ_Y (see, for instance, [16]). Assume that $P(B) = 0 \Leftrightarrow \mu(B) = 0$ for both $P \triangleq P_X$ and $P \triangleq P_Y$, where $B \in \mathcal{B}$. Both distribution functions F_X and F_Y are then strictly increasing and continuous surjective functions; hence, these functions are invertible. Let $\overline{X}$ denote an arbitrary random variable which induces Φ_X; it follows that a strictly increasing transformation $\hbar : \Omega \to \Omega$ exists, where $\hbar \triangleq F_Y^{-1} \circ F_X$, such that the random variable $\overline{Y} = \hbar \circ \overline{X}$ induces Φ_Y. Furthermore, the inverse function $\hbar^{-1} = F_X^{-1} \circ F_Y$ exists, and, for an arbitrary random variable $\overline{Y}$ that induces Φ_Y, there exists a random variable $\overline{X} = \hbar^{-1} \circ \overline{Y}$ which induces Φ_X. We observe that both the transformations $\hbar$ and $\hbar^{-1}$ only depend on the underlying probability spaces (i.e., the distribution functions), and not the particular random variables $\overline{X}$ and $\overline{Y}$.

IV. Novel Technique

We propose a novel technique for constructing distinct maps $\tilde{S} : \Omega \to \Omega$ each having at least one invariant PDF equal to a prescribed density f^*, where $f^* > 0$ almost everywhere. Let $f_X = f^*$, such that the probability measure P_X of Φ_X corresponds to the required invariant measure. Let X induce Φ_X. A sufficient condition for a candidate map $\tilde{S}$ to have at least one invariant density f_X is that the random variable $\tilde{S} \circ X$ induces the probability space Φ_X; this follows from the fact that both X and $\tilde{S} \circ X$ then have the PDF f_X.

The proposed technique first requires the selection of an arbitrary map $S : \Omega \to \Omega$ from the family of piecewise monotonic maps (we refer to this as *map pre-selection*). The function $Y = S \circ X$ is then a random variable which induces some probability space Φ_Y. We now consider a second transformation $\tau : \Omega \to \Omega$, and require that the function $\tau \circ Y$ corresponds to a random variable which induces the probability space Φ_X. It was stated in section III that there always exists a monotonic transformation $\hbar^{-1}$ from an arbitrary random variable which induces Φ_Y to another random variable which induces Φ_X. Therefore, we select $\tau \triangleq \hbar^{-1}$, such that $\hbar^{-1} = F_X^{-1} \circ F_Y$ transforms the random variable Y to some random variable $\hbar^{-1} \circ Y$ which induces Φ_X. Hence,

$$\tilde{S} = \hbar^{-1} \circ S = F_X^{-1} \circ F_Y \circ S \tag{5}$$

has at least one invariant probability density f_X.

Several observations regarding the map $\tilde{S}$ defined by (5) are relevant. First, the expression for $\tilde{S}$ may be rewritten by substituting (2) into (5), thereby obtaining an expression that only depends on the pre-selected map S and the distribution function F_X corresponding to the required invariant density. Second, since $\hbar^{-1}$ is strictly increasing, the map $\tilde{S}$ has the same number of branches and the same branch supports as the pre-selected map S, and preserves branch completeness. Hence, one may generate any number of structurally distinct maps which share at least one common invariant density f_X by selecting the branch characteristics of S. Third, whereas at least one invariant density of $\tilde{S}$ is equal to f_X, additional conditions pertaining to the map structure have to be derived in order to ensure that the invariant density is unique (i.e., such that $\tilde{S}$ is ergodic). We derive such conditions in the next section for a new family of piecewise monotonic maps.

V. Piecewise monotonic hat maps

In this section, we define the class of piecewise monotonic hat maps. We distinguish between *piecewise linear* and *piecewise nonlinear* maps from this class; with respect to the solution of the IFPP, pre-selection of S is done from the subset of piecewise linear maps, whereas the ergodic map $\tilde{S}$ (i.e., the IFPP solution) is generally found to be piecewise nonlinear.

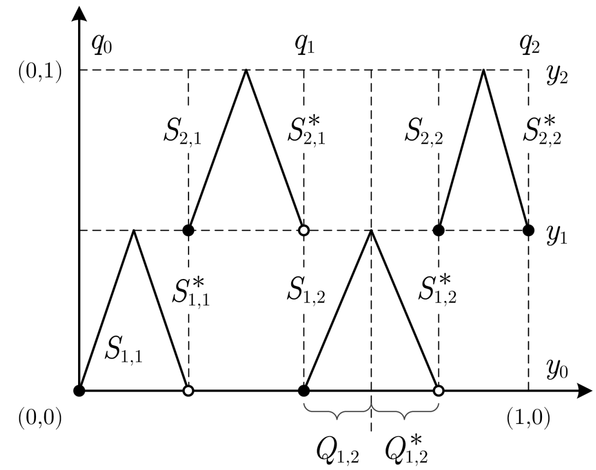

Fig. 2. Example of a 2×2 piecewise linear and monotonic hat map.

The $N \times N$ piecewise linear and monotonic hat maps (refer to Fig. 2 for an example), where $N > 1$, are defined by

$$S(x) = \sum_{n=1}^{N} \sum_{m=1}^{N} S_{n,m}(x) \mathbb{1}_{Q_{n,m}}(x) + S^*_{n,m}(x) \mathbb{1}_{Q^*_{n,m}}(x), \tag{6}$$

where $S_{n,m}(x) = 2\Delta_{n,m}(x - b_{n,m}) + y_{n-1}$, and where $S^*_{n,m}(x) = -2\Delta_{n,m}(x - b^*_{n,m}) + y_{n-1}$, and $\mathbb{1}_Q(x) \triangleq 1$ if $x \in Q$ and $\mathbb{1}_Q(x) \triangleq 0$ otherwise. The map parameters are related according to

$$b_{n,m} = q_{m-1} + \sum_{p=1}^{n-1} [y_p - y_{p-1}]/\Delta_{p,m}, \tag{7}$$

$$b^*_{n,m} = b_{n,m} + [y_n - y_{n-1}]/\Delta_{n,m}, \tag{8}$$

and $Q_{n,m} = [b_{n,m}, b_{n,m} + c_{n,m})$, $Q^*_{n,m} = [b_{n,m} + c_{n,m}, b^*_{n,m})$, where $c_{n,m} = [b^*_{n,m} - b_{n,m}]/2$, as well as

$$\sum_{n=1}^{N} (y_n - y_{n-1}) \Delta^{-1}_{n,m} = q_m - q_{m-1} \tag{9}$$

for $m = 1, 2, \ldots, N$, where $y_0 = q_0 \triangleq 0$ and $y_N = q_N \triangleq 1$. The proposed approach to selecting a piecewise linear hat map is to specify the parameters $\Delta_{n,m} > 1$ for $n = 1, 2, \ldots, N-1$ and $m = 1, 2, \ldots, N$, as well as the interval endpoints q_n and y_n for $n = 1, 2, \ldots, N-1$; the remaining parameters are then computed from (7) to (9).

A simplified expression for the distribution function F_Y, as required to derive $\tilde{S} = F_X^{-1} \circ F_Y \circ S$ in the case where S is a piecewise linear hat map, is obtained from (2) as

$$F_Y|_{(y_{n-1}, y_n)}(y) = \sum_{m=1}^{N} \left[C_{n,m}(y) + D_{n,m}(y) + \sum_{p=1}^{n-1} E_{p,m} \right], \tag{10}$$

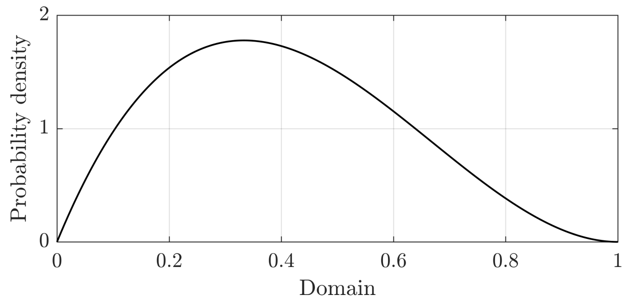

Fig. 3. The beta invariant PDF with parameters $\alpha = 2$ and $\beta = 2.5$.

where $C_{n,m}(y) \triangleq F_X \circ S^{-1}_{n,m}(y) - F_X(b_{n,m})$, $D_{n,m}(y) \triangleq F_X(b^*_{n,m}) - F_X \circ S^{*-1}_{n,m}(y)$, and $E_{p,m} \triangleq F_X(b^*_{p,m}) - F_X(b_{p,m})$.

Conditions under which $\tilde{S}$ belongs to the class of ergodic Markov maps are given by the Folklore theorem [17]. One of the requirements of this theorem is that $\tilde{S}$ possess the Markov property, which corresponds to the set of conditions

$$y_n = F_Y^{-1} \circ F_X(q_n), \;\; n = 1, 2, \ldots, N-1. \tag{11}$$

The map $\tilde{S}$ may be constructed iteratively to satisfy these conditions; specifically, the points q_n are specified beforehand for all n, whereas the points y_n are computed sequentially from (11) for $n = 1, 2, \ldots, N-1$ (when computing y_n, points $y_{n+1}, y_{n+2}, \ldots, y_{N-1}$ are not required to evaluate F_Y^{-1}).

The power spectrum of the map $\tilde{S}$ was found to be well approximated by the multimodal power spectrum of the pre-selected map S. The matrix representation $\mathbf{P}_S$ of the FP operator $\mathcal{P}_S$ (associated with S) has elements given by $\mathbf{P}_S = [\Delta^{-1}_{n,m}]^N_{n,m=1}$. It follows from [12] that the structure of piecewise linear hat maps leads to the center frequencies and bandwidths of N of the spectral modes being fully determined by the eigenvalues of $\mathbf{P}_S$, whereas the remaining N spectral modes no longer appear in the power spectrum. Hence, the hat map structure provides control over the characteristics of *all* spectral modes that are present, unlike previous solutions to the IFPP [10], [11]. More specifically, the spectral characteristics of $\tilde{S}$ may be prescribed by performing a numeric search for parameters $\Delta_{n,m}$ such that $\mathbf{P}_S$ has the required eigenvalues.

VI. Numerical Experiments

To illustrate the flexibility of the proposed approach in accommodating the simultaneous and independent specification of the invariant density and power spectrum properties, several ergodic hat maps $\tilde{S} = \hbar^{-1} \circ S$ having unimodal and bimodal power spectra with distinct characteristics were designed. The unimodal and bimodal spectra were selected to have variable mode bandwidths and mode center frequencies equal to 0 rad / sample and $\pm 2\pi/3$ rad / sample, respectively. The beta invariant PDF $f_X(x) \triangleq x^{\alpha-1}(1-x)^{\beta-1}/B(\alpha,\beta)$ was selected for all maps, with parameters $\alpha = 2$ and $\beta = 2.5$, where $B(\cdot)$ is a scaling factor which ensures $||f_X||_1 = 1$ (see Fig. 3).

Figure 4 is a plot of a sample map $\tilde{S}$ with the prescribed statistical properties. The map is observed to be piecewise nonlinear and incomplete. Figure 5 contains plots of the power spectra of all maps $\tilde{S}$ constructed using the proposed technique. The spectra were obtained via numerical iteration of the maps. The blue and purple curves correspond to the unimodal spectra, whereas the red and yellow curves correspond to the bimodal power spectra (the DC offset of the trajectories were removed). It is observed that the unimodal and bimodal nature of the power spectra become increasingly apparent as the mode bandwidth is decreased. The center frequencies of the modes are consistent with the prescribed frequencies.

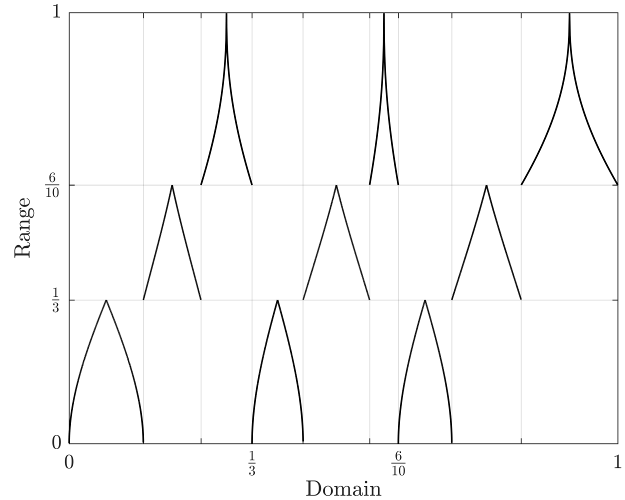

Fig. 4. A piecewise nonlinear hat map $\tilde{S}$ with the beta invariant PDF.

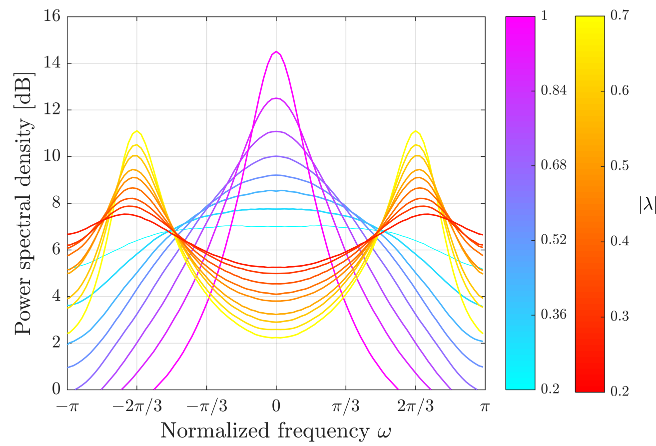

Fig. 5. Unimodal and bimodal power spectral densities of piecewise nonlinear hat maps $\tilde{S}$ sharing the beta invariant density, with different mode bandwidths. The bandwidths are determined by the prescribed eigenvalue magnitude $|\lambda|$, where $\lambda_1 = 1$ is the Perron eigenvalue, and $\lambda \triangleq \lambda_2 = \lambda_3^*$.

VII. Conclusion

The outcome of the numerical experiments leads to the conclusion that the proposed approach provides the freedom to simultaneously and independently prescribe the invariant density function and the properties of the map's power spectrum (i.e., the mode center frequencies and bandwidths). This allows for the construction of more realistic system and process models, as compared to existing solutions for the IFPP.

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